3
Frequency Domain and Filtering

3.1 Introduction

Comovement and volatility are key concepts in macroeconomics. Therefore, it is important to have statistics that describe the amount of volatility and the extend to which variables move together. The impulse response functions one obtains from structural Vector Autoregressive models (VARs) describe with how much each variable responds to structural shocks and how the variables morve together. One couldn't ask for more! The problem is that one needs assumptions to identify these impulse response functions and the value of the estimated impulse response functions depends on how realistic the assumptions of the structural VAR are.[1] A less ambitious approach to describe the comovement and volatilities is to calculate correlation coefficients and variances. The problem is that many macroeconomic series are not stationary, which means that moments are not defined.

Consequently, in order to calculate moments one has to at least first "extract" the non-stationary" part from the series. But even if data series are stationary, one may want to extract particular types of fluctuations out of the data. For example, if an economic model—say a model to explain business cycles—ignores seasonal fluctuations, then one would like to extract the seasonal component out of the data before one compares the

[1] Besides the identification assumption, one also has to take a stand on when to truncate the lag structure, since with a finite amount of data one can only estimate so many lags. Especially for long-run restrictions, the truncation may affect the results.

properties of the model with the properties of the data. Similarly, if the model is designed to model business cycle fluctuations, but is not meant to describe longer-term fluctuations, then one would like to extract these longer-term fluctuations from the data *and* from the data generated by the model. This way one compares that part of the observed data with that part of the generated data that the model was intended to model.

These informal statements can be made precise if one describes stochastic processes in the frequency domain instead of the time-series domain

3.2 Fouriers transforms

In this section, I give some definitions. Don't worry if these definitions look weird, but they turn out to be very useful concept. Consider an infinite sequence of absolutely summable Numbers, $\{\gamma_j\}_{j=-\infty}^{\infty}$. The fourier transform of this sequence is defined as

$$F_x(\omega) = \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j}. \tag{3.1}$$

The Riesz-Fischer theorem guarantees that the fourier transform exists (and that truncated versions of the spectrum converge towards it). If $\gamma_j = \gamma_{-j}$, then the fourier transform can also be written as

$$\begin{aligned} F_x(\omega) &= \gamma_0 + \sum_{j=1}^{\infty} \gamma_j (e^{-i\omega j} + e^{i\omega j}) \\ &= \gamma_0 + \sum_{j=1}^{\infty} 2\gamma_j \cos(j\omega) \end{aligned} \tag{3.2}$$

where the first equality follows from the fact that $\gamma_j = \gamma_{-j}$ and the last equality from Euler's formula which says you can write $e^{-i\omega j}$ as

$$e^{-i\omega j} = \cos(\omega j) - i \sin(\omega j). \tag{3.3}$$

Using the fourier transform, you can go from the autocovariances to the spectrum. But the Converse of the Riesz-Fischer theorem makes clear that you can also go from the fourier transform to the original numbers. That is,

$$\gamma_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_x(\omega) e^{i\omega h} d\omega. \tag{3.4}$$

Proof of the Converse of the Riesz-Fischer theorem

When we substitute the definition of the fourier transform into (3.4) we get

$$\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_j\int_{-\pi}^{\pi}e^{-i\omega j}e^{i\omega h}d\omega \tag{3.5}$$

Note that

$$\int_{-\pi}^{\pi}e^{-i\omega j}e^{i\omega h}d\omega = \int_{-\pi}^{\pi}e^{i\omega(h-j)}d\omega$$

$$\begin{cases} = \int_{-\pi}^{\pi}1d\omega = 2\pi \text{ if } h=j \\ = \int_{-\pi}^{\pi}\left(\cos(\omega(h-j)) + i\sin(\omega(h-j))\right)d\omega = 0 \text{ if } h\neq j \end{cases} \tag{3.6}$$

Using these in Equation (3.5) we get (3.4).■

3.3 Frequency versus Time-series domain

Thinking of a time series in the time-series domain is very natural. The value of a variable y_t is determined by factors that were already known in period $t-1$ and an innovation. That is,

$$x_t = \mathrm{E}\left[x_t|I_{t-1}\right] + \varepsilon_t, \tag{3.7}$$

where $\mathrm{E}[x_t|I_{t-1}]$ is the expectation of x_t conditional on information available in period $t-1$. An important building block of time-series analysis is the Wold decomposition that says that any covariance stationary stochastic process x_t can be written as

$$x_t = a + \sum_{j=0}^{\infty}b_j e_{t-j} + \eta_t, \tag{3.8}$$

where $b_0 = 1, \Sigma_{j=1}^{\infty}b_j^2 < \infty$, $\mathrm{E}e_t^2 = \sigma^2 \geq 0$, $\mathrm{E}e_t e_s = 0$ for $t \neq s$, $\mathrm{E}e_t = 0$, $\mathrm{E}\eta_t e_s = 0 \;\forall t, s$, and η_t is linearly deterministic. Make sure to understand the difference between ε_t in (3.7) and e_t in (3.8). The innovation ε_t is *independent* of any variables in the information set in period $t-1$, whereas the innovation e_t is only *not correlated* with any of the past innovations. But do not worry about the η_t term. That is just to be formal and hardly ever shows up in anything we use.

Now consider the fourier transform of a finite data set, $\{x_t\}_{t=1}^T$, scaled by $\sqrt{T}$

$$\tilde{x}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e^{-i\omega t} x_t. \tag{3.9}$$

Again, there is an inverse Fourier tranform. Let

$$\omega_j = (j-1)2\pi/T, \tag{3.10}$$

that is,the values of ω_j used are spread evenly across the unit circle. The *finite* inverse Fourier transform is given by

$$x_t = \frac{1}{\sqrt{T}} \sum_{\omega_j} e^{i\omega_j t} \tilde{x}(\omega_j). \tag{3.11}$$

You can find a proof in Cochrane's notes (Section 9.1). Define $\phi(\omega)$ implicitly by

$$\tilde{x}(\omega) = |\tilde{x}(\omega)|\, e^{i\phi(\omega)}. \tag{3.12}$$

The series x_t can now be written as

$$x_t = \frac{1}{\sqrt{T}} \left(\tilde{x}(0) + 2 \sum_{\omega_j < \pi} |\tilde{x}(\omega_j)| \cos(\omega_j t + \phi(\omega_j) \right) \tag{3.13}$$

That is, the data can be represented as a series of cosine waves of frequency ω_j that are *magnified* by $|\tilde{x}(\omega_j)|$ and shifted by $\phi(\omega_j)$. If ω_j is low then $\cos(\omega_j t)$ corresponds to waves with very long cycles and when ω_j is high then we get waves with very short cycles. Recall that the length of the cycle is called the period, and the period is equal to $2\pi/\omega$. The cosine waves with higher values of $|\tilde{x}(\omega_j)|$ are obviously more important for the total fluctuations in x_t.

The fact that a time series can be described by a weighted combination of "waves" with different frequencies, suggests two things. First, it suggests that we can determine which waves are more important, that is, we can determine whether waves with short cycles or waves with long cycles are more important for the behavior of x_t. Second, it suggests that we can extract that part of the series associated with the frequencies we are not interested in. The next two sections will make clear that these conjectures are correct.

3.4 The spectrum

Let x_t be a scalar covariance-stationary process with absolutely summable autocovariances and let γ_j be the j^{th} autocovariance. Note that γ_0 is the variance. Then the spectrum is defined as

$$S_x(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j}. \tag{3.14}$$

$\Sigma_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j}$ is the *Fourier* transform of the series $\{\gamma_j\}$. So, the spectrum is the Fourier transform of the autocovariances (divided by 2π). The Riesz-Fischer theorem guarantees that the spectrum exists (and that truncated versions of the spectrum converge towards it). The spectrum can also be written as

$$\begin{aligned} S_x(\omega) &= \tfrac{1}{2\pi}\gamma_0 + \sum_{j=1}^{\infty} \gamma_j (e^{-i\omega j} + e^{i\omega j}) \\ &= \tfrac{1}{2\pi}\gamma_0 + \sum_{j=1}^{\infty} 2\gamma_j \cos(j\omega) \end{aligned} \tag{3.15}$$

where the first equality follows from the fact that $\gamma_j = \gamma_{-j}$ and the last equality from Euler's formula which says you can write $e^{-i\omega j}$ as

$$e^{-i\omega j} = \cos(\omega j) - i \sin(\omega j). \tag{3.16}$$

Using the Fourier transform, you can go from the autocovariances to the spectrum. But the Converse of the Riesz-Fischer theorem makes clear that you can also go from the spectrum to the autocovariances. That is,

$$\int_{-\pi}^{\pi} S_x(\omega) e^{i\omega h} d\omega = \gamma_h. \tag{3.17}$$

Spectrum and variance

If we apply the converse of the Riesz-Fischer theorem for $h = 0$ we get

$$\int_{-\pi}^{\pi} S_x(\omega) d\omega = \gamma_0 \text{ or } 2\int_{0}^{\pi} S_x(\omega) d\omega = \gamma_0, \tag{3.18}$$

where the second expression follows from the fact that the spectrum is symmetric. That is, if we integrate the spectrum over all frequencies, then we get the variance. But note that we can also integrate the spectrum over some frequencies. For example, suppose that we are interested in determining how important business cycle fluctuations are for the volatility of x_t. Business cycle fluctuations are typically defined as those fluctuations that have a period (length of the cycle) that is less than 8 years. If one has quarterly data, then this would correspond with a period of less than 32 quarters or frequencies larger than $2\pi/32$. The variance of x_t that is due to business cycle frequencies, $\gamma_{0,bc}$ is thus equal to

$$\gamma_{0,bc} = 2 \int_{\pi/16}^{\pi} S_x(\omega) d\omega, \tag{3.19}$$

whereas the variance of x_t that is due to lower frequency cycles is equal to

$$\gamma_{0,non-bc} = 2 \int_{0}^{\pi/16} S_x(\omega) d\omega. \tag{3.20}$$

Note that the two variance measures add up to the variance of x_t.

Deeper stuff

How is the Spectrum related to the frequency domain representation of x_t given above in 3.13? Unfortunately, this is not that easy to show although the result that the spectrum integrates to the *total* variance already indicates that there is likely to be a link. In fact, it can be shown that $S_x(\omega)$ is equal to $|\tilde{x}(\omega)|$, that is the factor in the frequency domain representation that measure the importance of a particular cosine wave is exactly the value of the Spectrum.[2] That $|\tilde{x}(\omega)|$ is related to the sum of covariances is not that surprising if you realize that $\tilde{x}(\omega)$ is defined as the (weighted) sum of x_t. The value of $|\tilde{x}(\omega)|$ which is equal to $\tilde{x}(\omega)$ and its conjugate is thus related to the product of the two sums. This gives you all the autocovariances.

Easy way to find the spectrum

Consider the process

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j} = b(L) x_t. \tag{3.21}$$

Note that x_t could be any stationary process. Then the spectrum of y_t is given by

$$S_y(\omega) = \left| b(e^{-i\omega}) \right|^2 S_x(\omega), \tag{3.22}$$

where $\left| b(e^{-i\omega}) \right|$ is the norm of $b(e^{-i\omega})$. A proof is given in the appendix. For example, suppose that $x_t = \varepsilon_t$ is white noise. It is easy to calculate that the spectrum of a white noise process is equal to

$$S_\varepsilon(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}. \tag{3.23}$$

[2] You can find the proof in Cochrane.

If y_t is an AR(1) process, that is, $y_t = \rho_y y_{t-1} + \varepsilon_t$ or $y_t = (1-\rho_y L)^{-1}\varepsilon_t$, then

$$
\begin{aligned}
S_y(\omega) &= \left|\frac{1}{1-\rho_y e^{-i\omega}}\right|^2 \frac{\sigma_\varepsilon^2}{2\pi} && (3.24)\\
&= \left(\frac{1}{1-\rho_y e^{-i\omega}}\right)\left(\frac{1}{1-\rho_y e^{+i\omega}}\right)\frac{\sigma_\varepsilon^2}{2\pi} && (3.25)\\
&= \frac{1}{1-\rho_y(e^{+i\omega}+e^{-i\omega})+\rho_y^2}\frac{\sigma_\varepsilon^2}{2\pi} && (3.26)\\
&= \frac{1}{1-2\rho_y\cos\omega+\rho_y^2}\frac{\sigma_\varepsilon^2}{2\pi} && (3.27)
\end{aligned}
$$

If $z_t = (1-\rho_z L)^{-1}y_t$ one would get

$$S_z(\omega) = \frac{1}{1-2\rho_z\cos\omega+\rho_z^2}\frac{1}{1-2\rho_y\cos\omega+\rho_y^2}\frac{\sigma_\varepsilon^2}{2\pi} \tag{3.28}$$

Filters

In this section, we show how we can construct filters that extract that part of x_t that is associated with particular frequency bands. For example, we will show how to decompose x_t so that $x_t = x_t^{bc} + x_t^{non-bc}$, x_t^{bc} is the part that is associated with business cycle frequencies ($\omega \geq \pi/16$), and x_t^{non-bc} is the part that is associated with lower frequencies ($\omega < \pi/16$).

Filters have the following form

$$x_t^f = \sum_{j--\infty}^{\infty} b_j x_{t-j} = b(L)x_t. \tag{3.29}$$

Above, we showed that it is very easy to get the spectrum of x_t^f. In particular,

$$S_{x^f}(\omega) = \left|b(e^{-i\omega})\right|^2 S_x(\omega), \tag{3.30}$$

where $\left|b(e^{-i\omega})\right|$ is the norm of $b(e^{-i\omega})$ or the *gain* of the filter.

Consider the first-difference filter as an example. That is,

$$x_t^f = \Delta x_t = x_t - x_{t-1} = (1-L)x_t. \tag{3.31}$$

Then

$$S_{\Delta x}(\omega) = \left|(1-e^{-i\omega})\right|^2 S_x(\omega).$$

The term $\left|(1-e^{-i\omega})\right|^2$ is increasing in ω. In particular, $\left|(1-e^{-i\times 0})\right|^2 = 0$ and $\left|(1-e^{-i\times\pi})\right|^2 = 4$. Recall that the spectrum reveals the importance of different frequencies. Since for lower frequencies the gain of the filter is less than one, the first-difference filter reduces the importance of these frequencies. Since for higher frequencies the gain of the filter is bigger than one, the first-difference filter reinforces the importance of these frequencies. If a series is subject to a lot of measurement error, which is typically a high-frequency phenomenon, then it would not be a good idea to use the first-difference filter, since it would emphasize the measurement error.

Constructing the ideal filter

Suppose I would like to "take out" that part of the series x_t that is associated with frequencies between ω_1 and ω_2. This is called a band-pass filter, since everything in the band $[\omega_1, \omega_2]$ passes through the filter. Another way of saying the same thing is that I want

$$S_{x^f}(\omega) = S_x(\omega) \ \text{ if } \ \omega \in [\omega_1, \omega_2]$$

and

$$S_{x^f}(\omega) = 0 \ \text{ if } \ \omega \in [0, \omega_1) \cup (\omega_2, \pi].$$

Using the relationship in Equation (3.30), we can accomplish this if the gain of the filter satisfies

$$\left|b(e^{-i\omega})\right| = 1 \ \text{ if } \ \omega \in [\omega_1, \omega_2]$$

and

$$\left|b(e^{-i\omega})\right| = 0 \ \text{ if } \ \omega \in [0, \omega_1) \cup (\omega_2, \pi].$$

This would be accomplished if

$$b(e^{-i\omega}) = b(e^{+i\omega}) = 1 \ \text{ if } \ \omega \in [\omega_1, \omega_2]$$

and

$$b(e^{-i\omega}) = b(e^{+i\omega}) = 0 \ \text{ if } \ \omega \in [0, \omega_1) \cup (\omega_2, \pi].$$

Note that $b(e^{-i\omega})$ is the Fourier transform of the b_j coefficients.[3] But this means that if we know $b(e^{-i\omega})$, we can use the converse of the Riesz-Fischer

[3] That is, it is like a spectrum but we are only not dividing by 2π. Since $b(e^{-i\omega})$ is not a spectrum, we divide by 2π in the expression for b_j below.

theorem to back out the b_j coefficients. In particular, we have

$$\begin{aligned}
b_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b(e^{-i\omega})e^{+i\omega j}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \left[b(e^{-i\omega})e^{+i\omega j} + b(e^{+i\omega})e^{-i\omega j}\right] d\omega \\
&= \frac{1}{2\pi}\left(\int_{0}^{\omega_1} 0 d\omega + \int_{\omega_1}^{\omega_2} 1 \times \left[e^{-i\omega j} + e^{+i\omega j}\right] d\omega + \int_{0}^{\pi} 0 d\omega\right) \\
&= \frac{1}{\pi}\int_{\omega_1}^{\omega_2} \cos(\omega j) d\omega \\
&= \frac{1}{\pi}\left(\left.\frac{\sin(\omega j)}{j}\right|_{\omega_1}^{\omega_2}\right).
\end{aligned}$$

This gives

$$b_j = \frac{\sin\omega_2 j - \sin\omega_1 j}{\pi j} \tag{3.32}$$

To evaluate b_j you have to use L'Hopital's rule and you get

$$b_0 = \frac{\omega_2 - \omega_1}{\pi} \tag{3.33}$$

Note that the only inputs of our filtering process are the two boundary values ω_1 and ω_2. For these two inputs we can easily calculate the coefficients of the band-pass filter. Note that ω_1 could be 0 in which case the filter is also called a low-pass filter and ω_2 could be equal to π in which case the filter is also called a high-pass filter.

There is only one problem and that is that the filter is an infinite-order two-sided filter. If you want to apply this filter to a data series with finite length, we have to truncate it. Below, we will show how to do this, but before we do, we make a short digression and talk about integrated processes.

I(1) processes

We know that many macroeconomic processes are I(1) processes. A process y_t is I(1) if $y_t - y_{t-1}$ is stationary. Let $y_t = \Sigma_{j=0}^{\infty} a_j \varepsilon_{t-j} = a(L)\varepsilon_t$, where ε_t is white noise. For an integrated process, $a(1)$, i.e., the infinite sum of the a_j coefficients, is infinite.

Above, we defined the spectrum for stationary processes, but that doesn't mean that the analysis above doesn't apply (without some modification) to

I(1) processes. One must be a little bit careful though. Consider the process $y_{\rho,t}$ that satisfies

$$y_{\rho,t} = \frac{1}{1-\rho L} x_t \tag{3.34}$$

If x_t is a stationary process and $|\rho| < 1$ then $y_{\rho,t}$ is a stationary process. Consequently, the spectrum of $y_{\rho,t}$ is well defined. In particular,

$$S_{\rho,y}(\omega) = \frac{1}{1-2\rho\cos\omega+\rho^2} S_x(\omega) \tag{3.35}$$

If ρ is equal to 1, then $y_{\rho,t}$ is an I(1) variable. Moreover, if $\rho = 1$ and $\omega = 0$ then the spectrum clearly is not well-defined. Note however that for all other values of ω the spectrum is well defined even if $\rho = 1$. In other words, the spectrum of an I(1) process goes to infinity as ω goes to zero. This is very intuitive. It basically says that cycles with an infinite cycle (i.e. permanent effects) get an infinite weight.

Now suppose that we have an I(1) process y_t and we apply the filter $b(L)$. What do we know about the spectrum of the filtered series $b(L)y_t$? We know that we can write an I(1) process as

$$y_t = \frac{1}{1-L} x_t,$$

where x_t is a stationary process. If it was the case that our filter $b(L)$ could be written as

$$\begin{aligned} b(L) &= (1-L)\overline{b}(L) \\ \text{with } \overline{b}(1) &< \infty \end{aligned}$$

then we know that our filtered series is stationary? Why? Just combine the formulas.

$$\begin{aligned} y_t^f &= b(L)y_t \\ &= (1-L)\overline{b}(L)y_t \\ &= \frac{(1-L)\overline{b}(L)}{1-L} x_t \\ &= \overline{b}(L)x_t \end{aligned}$$

When is it possible to write $b(L)$ as

$$b(L) = (1-L)\overline{b}(L)$$

with $\overline{b}(1) < \infty$? This would be the case if $L = 1$ is a root of $b(L)$. That is, we need $b(1) = 0$. So, if a filter has the property that $b(1) = 0$, then if would make a non-stationary series stationary. One obvious filter that has this property is the first-difference filter, $1 - L$.

Constructing a practical filter.

Above we derived the formulas for our band pass filter. The problem is that it was a two-sided infinite-order filter, so we have to truncate the filter. Fortunately, the coefficients approach zero, but since they wouldn't be zero at any finite lag (lead) we do make an approximation error. One obvious approach would be to simply use

$$x_t^f = \overline{b}(L)x_t = \sum_{j=-J}^{J} \overline{b}_j x_{t-j} \text{ with } \overline{b}_j = b_j \tag{3.36}$$

But from the discussion in the last subsection, we know that it would be nice if the practical filter $\overline{b}(L)$ has the property that $\overline{b}(1) = 0$. If true, then it would transform I(1) processes into stationary processes. The ideal filter has this property, but the truncated does not necessarily have this property. So besides the truncation, we also have to do a minor correction to ensure that $\overline{b}(1) = 0$, that is, we have to ensure that the coefficients of $\overline{b}(L)$ add up to zero. But this is easy to do. Let

$$\mu_b = \frac{-\sum_{j=-J}^{J} b_j}{2J+1}$$

Now define the practical filter as

$$x_t^f = \overline{b}(L)x_t = \sum_{j=-J}^{J} \overline{b}_j x_{t-j} \text{ with } \overline{b}_j = b_j + \mu_b. \tag{3.37}$$

3.5 Hodrick-Prescott filter

A very popular filter in macroeconomics is the Hodrick-Prescott filter. For a process x_t it defines a trend term $x_{\tau,t}$. The cyclical term is then given by $x_{c,t} = x_t - x_{\tau,t}$. The trend term is defined as follows

$$\min_{\{x_{\tau,t}\}_{t=1}^{T}} \sum_{t=2}^{T-1} (x_t - x_{\tau,t})^2 + \lambda \sum_{t=2}^{T-1} \left\{ \left[(x_{\tau,t+1} - x_{\tau,t}) - (x_{\tau,t} - x_{\tau,t-1}) \right]^2 \right\}$$

with $\lambda > 0$. The first terms says that it helps the optimization if the trend, $x_{\tau,t}$ is a good fit for the actual series. The second term says that from one period to the next the change in the trend term cannot fluctuate too much. If you would set $x_{\tau,t} = x_t$ then you set the first term equal to zero, but then the trend term is likely to fluctuate a lot, which means that the second term, i.e., the penalty term is high.

For quarterly data it is common to use $\lambda = 1600$. It turns out that this filter is very close to a low-pass filter that lets through all frequencies less than $\pi/16$. This means that the residual $x_t - x_{\tau,t}$ would correspond to business cycle frequencies.

3.6 Appendix

3.6.1 *Spectrum of* $y_t = b(L)x_t$

The k^{th} autocovariance of y_t, $\gamma_k(y)$ is given by

$$\gamma_k(y) = \mathrm{E}\left[y_t y_{t-k}\right] = \mathrm{E}\left[\sum_{j_1} b_{j_1} x_{t-j_1} \sum_{j_2} b_{j_2} x_{t-k-j_2}\right]$$
$$= \sum_{j_1,j_2} b_{j_1} b_{j_2} \mathrm{E}\left[x_{t-j_1} x_{t-k-j_2}\right] = \sum_{j_1,j_2} b_{j_1} b_{j_2} \gamma_{j_1-k-j_2}(x).$$

Let $h = k + j_2 - j_1$. Then

$$\begin{aligned}
S_y(\omega) &= \frac{1}{2\pi} \sum_k \gamma_k(y) e^{-i\omega k} \\
&= \frac{1}{2\pi} \sum_{k,j_1,j_2} e^{-i\omega k} b_{j_1} b_{j_2} \gamma_{j_1-k-j_2}(x) \\
&= \frac{1}{2\pi} \sum_{h,j_1,j_2} e^{-i\omega(h+j_1-j_2)} b_{j_1} b_{j_2} \gamma_{-h}(x) \\
&= \frac{1}{2\pi} \sum_{h,j_1,j_2} e^{-i\omega(h+j_1-j_2)} b_{j_1} b_{j_2} \gamma_h(x) \\
&= \frac{1}{2\pi} \sum_{j_1} e^{-i\omega j_1} b_{j_1} \sum_{j_2} e^{+i\omega j_2} b_{j_2} \sum_h e^{-i\omega h} \gamma_h(x) \\
&= b(e^{-i\omega}) b(e^{+i\omega}) S_x(\omega) \\
&= \left| b(e^{-i\omega}) \right|^2 S_x(\omega)
\end{aligned}$$