Overlapping Generations Models

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Idea behind OLG models

- Introduce life-cycle aspects into macro models. Important for many questions:
  - pensions and retirement
  - savings
  - models with housing
  - models with education
- Basic framework:
  - fixed life duration
  - each period a new generation is born

Standard growth model in OLG framework

- each generation lives for 30 years
- elastic supply of 1 unit of labor
- household owns the capital stock
- firms hire labor at rate w_t and capital at rate r_t
- competitive markets

Standard growth model in OLG framework

$$V_\tau(k_{\tau,t}, z_t; s_t) = \max_{k_{\tau+1,t+1}} \ln(w_t + r_t k_{\tau,t} - k_{\tau+1,t+1}) + \beta V_{\tau+1}(k_{\tau+1,t+1}, z_{t+1}; s_{t+}$$

- s_t: other state variables
- FOCs:
  - look just like those of representative agent model
  - except for $\tau = 30$ because $k_{31,t} \geq 0$ is now a binding constraint
- Firm problem unchanged

Equilibrium conditions

- Supply

$$K_t = \sum_{\tau=1}^{30} k_{\tau,t}$$

$$H_t = \sum_{\tau=1}^{30} 1 = \bar{H} = 30$$

- Demand

$$r_t = \alpha z_t \left(k_{j,t} / h_{j,t}\right)^{\alpha-1}, \quad w_t = (1-\alpha) z_t \left(k_{j,t} / h_{j,t}\right)^{\alpha}$$

- Equilibrium

$$r_t = \alpha z_t \left(K_t / \bar{H}\right)^{\alpha-1}, \quad w_t = (1-\alpha) z_t \left(K_t / \bar{H}\right)^{\alpha}$$

Overaccumulation of capital

- Overaccumulation: when you can (i) reduce the capital stock and (ii) increase consumption in every time period
- Can not happen in standard representative agent model
  - if you could you clearly would not be at an optimum

Capital decision in representative agent framework

- Consider perfect-foresight version

$$k_t < k_{ss} \implies k_{t+1} > k_t$$

$$k_t > k_{ss} \implies k_{t+1} < k_t$$

- Showing that you want to decrease (increase) capital when $k_t > k_{ss}$ ($k_t < k_{ss}$) is not difficult but requires some work
- Proof is *trivial* though when $k_t > k^{gr}$ (Golden Rule capital stock)

Golden Rule capital stock

- Hypothetical optimization problem: choose highest possible steady state (ignoring initial condition for capital)

$$\max_{c,k} c$$
$$c + k = k^{\alpha} + (1-\delta)k$$

First-order condition

$$k^{gr} = \left(\frac{\delta}{\alpha}\right)^{1/(\alpha-1)}$$

Golden Rule capital stock

Consider the actual infinitely-lived representative agent economy with initial capital stock k_1.

- Suppose $k_1 = k^{gr}$
  - Not optimal to keep capital constant and remain at highest possible steady state consumption level.
  - Decreasing capital leads to higher consumption now and lower consumption later. You are better off but showing this requires a bit of work
- Suppose $k_1 > k^{gr}$
  - Trivial to show that it is not optimal to keep capital at k_1
- In contrast, in an OLG model the economy's capital stock can remain at levels above k^{gr}

Non-monetary OLG model

Problem of the individual

$$\max_{c_t^y, c_{t+1}^o, s_{t+1}} U(c_t^y, c_{t+1}^o)$$
$$\text{s.t.} \quad c_t^y + s_{t+1} = 1,$$
$$c_{t+1}^o = (1 + r_{t+1})s_{t+1},$$

First-order condition:

$$\frac{\partial U(c_t^y, c_{t+1}^o)}{c_t^y} = \frac{\partial U(c_t^y, c_{t+1}^o)}{c_{t+1}^o}(1 + r_{t+1})$$

Equilibrium in an OLG model

- Equilibrium is more than just checking equations
- Also requires thinking about how to save and implementation
- Suppose no storage technology
  - only equilibrium is autarky: $c_t^y = 1$ and $c_t^o = 0$ for all t
- Suppose there are bonds
  - doesn't help

Pareto improvement in non-monetary OLG model

- Consider the autarky solution
  - reducing c_t^y with ε and increasing c_{t+1}^o (from zero) with ε increases utility for sure for small ε.
- This cannot be implemented in a competitive equilibrium
- Social planner could imprement this

Non-monetary OLG with storage

Suppose agent can store at rate r. ($1 + r > 0$ but r could be negative)

- Young (i.e. economy) will save even if $r < 0$ (for standard preferences)
- This is not PO (and like overaccumulation of capital)
- With population growth a CE is not PO when $r < n$

Non-monetary OLG with production technology

$$\max_{c_t^y, c_{t+1}^o, k_{t+1}} U(c_t^y, c_{t+1}^o)$$

$$\text{s.t.} \quad c_t^y + k_{t+1} = 1$$

$$c_{t+1}^o = k_{t+1}^\alpha + (1-\delta) k_{t+1}$$

Non-monetary OLG with production technology

- First-order condition OLG model

$$\frac{\partial U(c_t^y, c_{t+1}^o)}{\partial c_t^y} = \frac{\partial U(c_t^y, c_{t+1}^o)}{\partial c_{t+1}^o} \left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right).$$

Steady state:

$$\frac{\partial U(c^y, c^o)}{\partial c^y} = \frac{\partial U(c^y, c^o)}{\partial c^o} \left(\alpha k^{\alpha-1} + 1 - \delta\right) \quad (1)$$

- Corresponding equation in model with infinitely-lived agent:

$$\frac{\partial U(c)}{\partial c} = \left(\alpha k^{\alpha-1} + 1 - \delta\right) \beta \frac{\partial U(c)}{\partial c} \text{or}$$

$$1 = \left(\alpha k^{\alpha-1} + 1 - \delta\right) \beta. \quad (2)$$

- Solution to (2) always below k^{gr}
- Solution to (1) could be above k^{gr} so that

$$r_{t+1} = \alpha k_{t+1}^{\alpha-1} - \delta < 0$$

Monetary OLG model without storage

$$\max_{c_t^y, c_{t+1}^o, M_t^d} u(c_t^y, c_{t+1}^o)$$
$$\text{s.t.} \quad M_t^d = p_t(1 - c_t^y)$$
$$p_{t+1} c_{t+1}^o = M_t^d + T_{t+1}$$

First-order condition household

$$\frac{\partial u(c_t^y, c_{t+1}^o)}{\partial c_t^y} = \frac{\partial u(c_t^y, c_{t+1}^o)}{\partial c_{t+1}^o} \frac{p_t}{p_{t+1}} \tag{3}$$

Government

$$M_t^s - M_{t-1}^s = (1+n)^{t-1} T_t.$$

Monetary OLG model; Equilibrium condition

Money market equilibrium:

$$M_t^s = (1+n)^t M_t^d.$$

which implies (because of ...) equilibrium on the commodities market

$$(1+n)^t c_t^y + (1+n)^{t-1} c_t^o = (1+n)^t \times 1 \text{ or}$$

$$(1+n) c_t^y + c_t^o = (1+n) \times 1$$

- How many equations in how many unknowns?
- How many equilibria?

Monetary OLG model; Steady State

Money supply grows at constant rate $\mu \Longrightarrow$ Money market equilibrium:

$$M_t^s = (1+\mu)^t M_0^s = (1+n)^t M_t^d.$$

- In a steady state inflation rates are constant
- Money demand is only a function of inflation rate $\Longrightarrow$ *real* money demand, L_t, is constant, thus, $L_t = L_{t+1}$

Monetary OLG model; Steady State

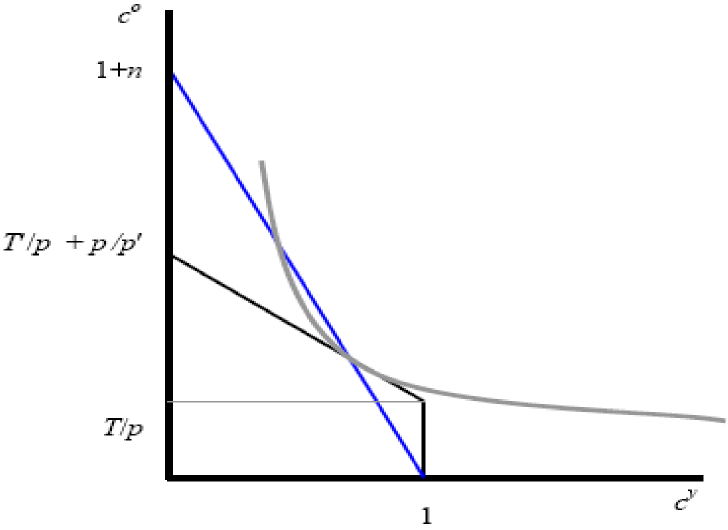
$$1 = \frac{L_t}{L_{t+1}} = \frac{M_t^d / p_t}{M_{t+1}^d / p_{t+1}}$$

$$= \frac{\frac{(1+\mu)^t M_0^s}{(1+n)^t}}{\frac{(1+\mu)^{t+1} M_0^s}{(1+n)^{t+1}}} \frac{p_{t+1}}{p_t} = \frac{(1+n)}{(1+\mu)} \frac{p_{t+1}}{p_t}$$

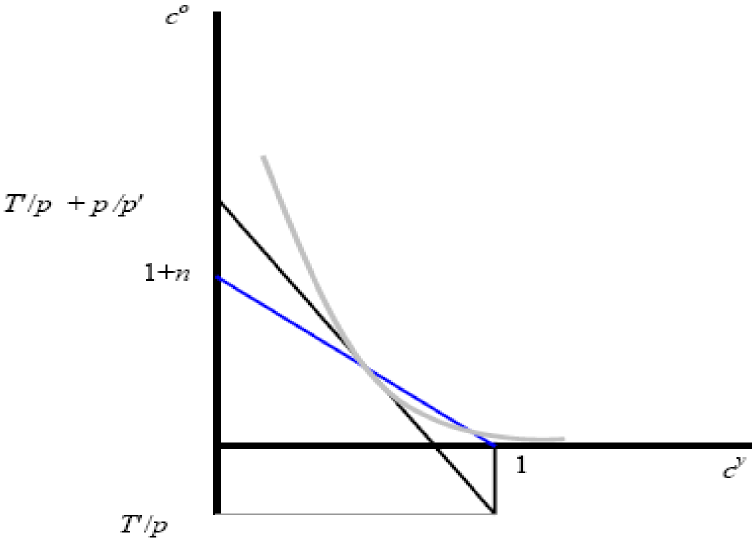
Steady state inflation is thus equal to

$$\frac{p_t}{p_{t+1}} = \frac{1+n}{1+\mu}.$$

Monetary equilibrium; positive money growth



Monetary equilibrium; negative money growth



Monetary OLG model; Social Planner

Problem if social planner gives equal weight to each generation:

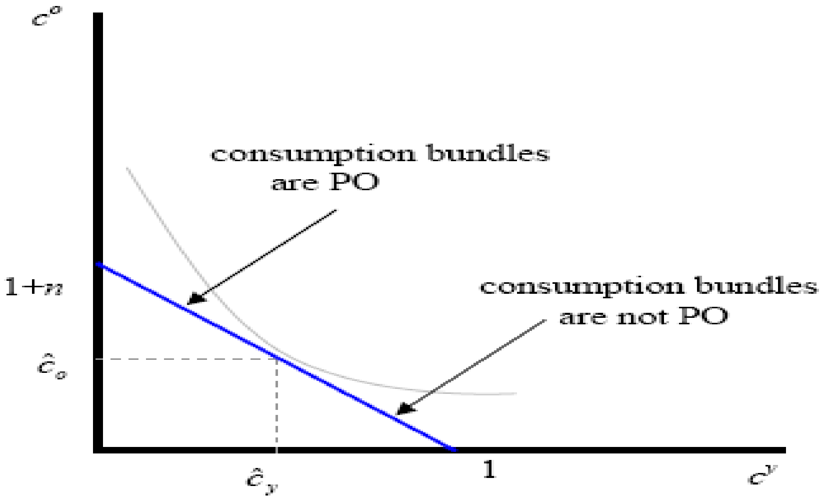
$$\max_{\{c_t^y, c_t^o\}_{t=1}^{\infty}} u(c_0^y, c_1^o) + \sum_{t=1}^{\infty} u(c_t^y, c_{t+1}^o)$$
$$\text{s.t.} \quad (1+n)c_t^y + c_t^o = 1+n$$

The Euler equation for this problem is given by

$$(1+n)\frac{\partial u(c_{t-1}^y, c_t^o)}{\partial c_t^o} = \frac{\partial u(c_t^y, c_{t+1}^o)}{\partial c_t^y}.$$

Denote solution by $\widehat{c}^o$ and $\widehat{c}^y$

Monetary OLG model; Muliple PO solutions



Monetary OLG model; CEPO

Social planner's steady state solution coincides with CE if

$$\frac{p_t}{p_{t+1}} = (1+n).$$

thus when $\mu = 0$.

Monetary OLG model; assumptions

$$v(c_t^y, c_{t+1}^o) = \frac{\partial U(c_t^y, c_{t+1}^o)/\partial c_t^y}{\partial U(c_t^y, c_{t+1}^o)/\partial c_{t+1}^o}.$$

- $\partial U(c_t^y, c_{t+1}^o)/\partial c_t^y) > 0$, $\partial U(c_t^y, c_{t+1}^o)/\partial c_{t+1}^o > 0$,
- Both consumption commodities are normal goods,
- $v(c_t^y, c_{t+1}^o)$ is continuous,
- $\lim_{c_t^y \to 0} v(c_t^y, c_{t+1}^o) = \infty$, and
- $\lim_{c_{t+1}^o \to 0} v(c_t^y, c_{t+1}^o) = 0$.

Monetary OLG model; CEPO

Proposition

If $\mu > 0$ *the steady-state monetary equilibrium is not Pareto optimal and if* $\mu \leq 0$ *the steady-state monetary equilibrium is Pareto optimal.*

Monetary OLG model; CEPO - intuition

Start at the point where $\mu = 0$ and thus $c^o = \widehat{c}^o$ & $c^y = \widehat{c}^y$. Now increase μ

- If $\mu \uparrow$ then $T \uparrow$ and $p/p' \downarrow$
- Under our regularity assumptions we get
  - $T \uparrow$, that is, the individual's budget constraint shifts out $\Longrightarrow c^y \uparrow$ and $c^o \uparrow$
  - $p/p' \downarrow$, that is, consumption when old gets more expensive $\Longrightarrow c^y \uparrow$ and $c^o \downarrow$
- Thus you get to a point where $c^y > \widehat{c}^y$ and $c^o < \widehat{c}^o$, i.e. a non PO point
- Similarly, if $\mu \downarrow$ then you get to a PO point

Monetary OLG model with storage

Proposition (existence of monetary equilibrium)

At least one monetary equilibrium exist if and only if $(1+n)/(1+\mu) \geq 1+r$.

Intuition: The return on money (LHS) has to be at least as high as the return on storage (RHS)

Monetary OLG model; existence - proof

Necessary part: Suppose to the contrary that

$$\frac{(1+r)(1+\mu)}{(1+n)} > 1$$

For agents to hold money you need

$$\frac{p_t}{p_{t+1}} \geq 1+r$$

Simple algebra gives

$$\frac{p_t}{p_{t+1}} = \frac{M^s_{t+1}}{(1+\mu)M^s_t}\frac{p_t}{p_{t+1}} = \frac{(1+n)M^d_{t+1}}{(1+\mu)M^d_t}\frac{p_t}{p_{t+1}} = \frac{(1+n)m_{t+1}}{(1+u)m_t}$$

Thus m_{t+1} grows without bound and thus at some point no longer feasible

Monetary OLG model; existence - proof

Sufficiency part: Agent's FOC:

$$v(1 - m_t, m_{t+1}(1+n)) = \frac{m_{t+1}}{m_t}\frac{1+n}{1+\mu}$$

Consider the equilibrium with $m_t = m\ \forall t$. Equilibrium requires

1. There is an m such that

$$v(1 - m, m(1+n)) = \frac{1+n}{1+\mu}$$

2. Return on money exceeds return on storage

$$\frac{m_{t+1}}{m_t}\frac{1+n}{1+\mu} = \frac{1+n}{1+\mu} \geq 1 + r$$

Monetary equilibria model; n>r

- non-monetary equilibrium clearly not PO
- monetary equilibria: storage not used so like case discussed above without storage

Monetary equilibria model; n<r

- non-monetary equilibrium clearly PO
- monetary equilibria can exist (need $\mu < 0$). If it exists it is PO

Monetary OLG model with storage