Solving Models with Heterogeneous Agents : Limited History Dependence

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Perturbation and employment shocks

- Suppose agents are subject to idiosyncratic unemployment shocks
    - $\varepsilon_{i,t} \in \{0, 1\}$ or $\varepsilon_{i,t} \in \{u, e\}$
- Could you solve such models using perturbation methods?

Perturbation and employment shocks

- To simplify discussion: no aggregate shocks
- FOCs:

for employed

$$c_{i,t} + k_{i,t} = (1 + r - \delta)k_{i,t-1} + w$$
$$c_{i,t}^{-\gamma} = \beta \mathbb{E}_t \left[c_{i,t-1}^{-\gamma}(1 + r - \delta) \right]$$

for unemployed

$$c_{i,t} + k_{i,t} = (1 + r - \delta)k_{i,t-1} + b$$
$$c_{i,t}^{-\gamma} = \beta \mathbb{E}_t \left[c_{i,t-1}^{-\gamma}(1 + r - \delta) \right]$$

Perturbation and employment shocks

Why couldn't we simply give the following model to Dynare?

for employed

$$c_{e,t} + k_{e,t} = (1 + r - \delta)k_{e,t-1} + w$$
$$c_{e,t}^{-\gamma} = \beta \mathbb{E}_t \left[c_{t-1}^{-\gamma}(1 + r - \delta) \right]$$

for unemployed

$$c_{u,t} + k_{i,t} = (1 + r - \delta)k_{u,t-1} + b$$
$$c_{u,t}^{-\gamma} = \beta \mathbb{E}_t \left[c_{t-1}^{-\gamma}(1 + r - \delta) \right]$$

variables

$$c_{e,t}, c_{u,t}, k_{e,t}, k_{u,t}$$

1. Typically we use borrowing constraints to keep problem well defined, but we could use smooth penalty functions instead.
2. What is the more fundamental problem?

Perturbation and employment shocks

- Koen Vermeylen of the University of Amsterdam was (I think) the first to realize this could be done. Vermeylen (2006) uses system of previous slide:
    1. keep track of *both* $k_{e,t}$ and $k_{u,t}$ *for all t*
    2. uses a well-chosen AR(1) process, z_t, that
        1. in simulation, shocks are such that $z_t \in \{0, 1\}$,
        2. selects current employement status, *and*
        3. the actual currentcapital stock:
           $k_{t-1} = (1 - z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}$.
    3. Substitute out k_{t-1}
- See appendix for details

LeGrand-Ragot (LGR) environment

- Exogenous aggregate risk affects rental rate of capital and wage rate
- Exogenous aggregate risk does *not* affect employment risk
  - but this can be done (as shown at end of slides)
- Incomplete markets
  - short-sell constraint and saving only through capital
  - *some* joint risk sharing as discussed below
- Preference shocks to get realistic wealth distribution
- An unemployed worker works δ hours at home to produce δ goods (parameters are chosen such that agents do not prefer to work less than δ)

Key approximating assumption

Key approximation step: All agents with the same employment history for the last N periods are identical

- If $N = 2$, then there are 4 types:
  uu, ue, eu, ee
- If $N = 3$, then there are 8 types:
  uuu,uue,ueu,uee,euu,eue,eeu,eee
  - (in general, if there are E individual states then there are $(E+1)^N$ types; here $E = 2$)
- Original model: $N = \infty$, that is, an infinite number of different agents

Stories representing approximation

- LGR propose two "stories/models" so that the set of equations given to the computer looks like *an actual economy* and not just an approximation to the original model
  1. quasi-planner
  2. decentralized version with particular insurance mechanism
- This is useful, for example, to understand whether the set of equations of the approximation is well behaved

Quasi planner "story"

- Agents with the same employment history for the last N periods have the same consumption and make the same savings choice *independent of the wealth they bring into period t*
- This savings choice is made by the quasi-planner
- The quasi-planner does take prices as given (in contrast to the conventional social planner)

Quasi-planner "story"

- Beginning of period t :
  - all agents with the same N-period employment history go to the same "island"
  - their savings are pooled
  - quasi-planner chooses consumption and savings
- End of period t : All agents are entitled to an equal share of the savings
  - Thus, quasi-planner cannot condition on *next-period's* unemployment status. This mimics market incompleteness

Quasi-planner model

$$
\begin{aligned}
& \max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \varepsilon^N} S_{t,e^N} \xi_{e^N} U\left(c_{t,e^N}, l_{t,e^N}\right)\right] \\
& \text{s.t.} \\
a_{t,e^N} + c_{t,e^N} &= w_t l_{t,e^N} n_{t,e^N} + \delta 1_{e^N=0} + \left(1 + r_t\right) \widetilde{a}_{t,e^N} \quad \forall e^N \in \varepsilon^N \\
a_{t,e^N} &\geq 0 \quad \forall e^N \in \varepsilon^N \\
\widetilde{a}_{t,e^N} &= \sum_{\widetilde{e}^N \in \varepsilon^N} \frac{S_{t-1,\widetilde{e}^N}}{S_{t,e^N}} \Pi_{t-1,(\widetilde{e}^N, e^N)} a_{t-1,\widetilde{e}^N} \quad \forall e^N \in \varepsilon^N \\
S_{t+1,e^N} &= \Pi_t S_{t,e^N} \\
l_{t,e^N} &\geq 0
\end{aligned}
$$

Quasi-planner model

- Index to indicate a particular type: $e^N \in \varepsilon^N$
- S_{t,e^N} : population size island e^N
- $\widetilde{a}_{t,e^N}$: per capita beginning-of-period wealth on island e^N
  - $S_{t,e^N}\widetilde{a}_{t,e^N}$ equals sum of savings brought to island e^N from different islands
- $1_{e^N=0}$: indicator function if agents on this island are unemployed
- n_{t,e^N} : idiosyncratic productivity agents on island e^N
  $n_{t,e^N} = 0$ if $1_{e^N=0} = 1)$
- ξ_{e^N} : preference parameter
  - agents with different employment histories have a different utility function
- Π_t : transition matrix for the full N-period employment state
  - examples below

Quasi-planner FOCs

$$\xi_{e^N} U_c\left(c_{t,e^N}, l_{t,e^N}\right) + \nu_{t,e^N}$$

$$=$$

$$\beta \mathbb{E}_t \left[\sum_{\widehat{e}^N \in \varepsilon^N} \Pi_{t,(e^N,\widehat{e}^N)} \xi_{\widehat{e}^N} U_c\left(c_{t+1,e^N}, l_{t+1,e^N}\right)\left(1 + r_{t+1}\right) \right]$$

$$\nu_{t,e^N} a_{t,e^N} = 0,\ a_{t,e^N} \geq 0,\ \nu_{t,e^N} \geq 0$$

$$w_t n_{e_t^N} U_c\left(c_{t,e^N}, l_{t,e^N}\right) = -U_l\left(c_{t,e^N}, l_{t,e^N}\right) \text{ if } n_{t,e^N} > 0$$

$$l_{t,e^N} = \delta \text{ if } n_{t,e^N} = 0$$

Quasi-planner FOCs

- Note that the population sizes drop out!
  - going to a large S_{t,e^N} island is *bad* because you have to share your wealth with more agents
  - going to a large S_{t,e^N} island is *good* because the social planner gives it a larger weight
  - these effecs exactly offset each other
- Note that the *linearized* Euler equation captures precautionary savings

Other model equations

aggregate labor supply
$$L_t = \sum_{e^N \in \varepsilon^N} S_{t,e^N} n_{t,e^N} l_{t,e^N}$$

aggregate savings
$$L_t = \sum_{e^N \in \varepsilon^N} S_{t,e^N} a_{t,e^N} = \sum_{e^N \in \varepsilon^N} S_{t+1,e^N} \widetilde{a}_{t+1,e^N}$$

wage rate
$$w_t = (1-\alpha) A_{t-1} \left(\frac{K_{t-1}}{L_t} \right)^{\alpha}$$

rental rate
$$r_t = \alpha A_{t-1} \left(\frac{K_{t-1}}{L_t} \right)^{\alpha - 1} - depreciation$$

productivity
$$A_t = 1 + u_t$$
$$u_t = \rho u_{t-1} + e_t$$

Specific assumptions

- Greenwood, Hercowitz, Huffman preferences

$$U_c\left(c_{t+1,e^N}, l_{t+1,e^N}\right) = \frac{\left(c_{t+1,e^N} - \frac{l_{t+1,e^N}^{1+1/\phi}}{1+1/\phi}\right)^{1-\gamma} - 1}{1-\gamma}$$

- $\Longrightarrow$ first-order condition for employed becomes

$$w_t n_{e_t^N} = l_{t,e^N}^{1/\phi}$$

Specific assumptions

- If $n_{e_t^N}$ (just as aggregate productivity) is known in period t, then L_t is known in period $t \implies r_t$ is known in period t (risk-free r_t means capital would be perfect substitute to risk-free government bonds)
- In fact, it is assume that $n_{e_t^N} = 1$ for all employed agents
- Π_t is constant $\implies$ unemployment rate is constant
- $N = 4$

Constructing transition matrix

16 groups:

unemployed	employed
1. uuuu	9. euuu
2. uuue	10. euue
3. uueu	11. eueu
4. uuee	12. euee
5. ueuu	13. eeuu
6. ueue	14. eeue
7. ueeu	15.eeeu
8. ueee	16.eeee

- probability to become employed for unemployed equals 0.5
- probability to become unemployed for employed equals 0.2

$$\Pi = \begin{bmatrix}
.5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 \\
.5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8
\end{bmatrix}$$

The tricky bit

- You have to figure out by trial and error (and some economic thinking) which group will be at the constraint
- Things would be problematic if that depends on the aggregate state
  - (less likely to be the case if aggregate fluctuations are small)
- Here, only group 1 turns out to be at the constraint

Some Dynare equations

- Budget constraint for group 1, uuuu, thus currently unemployed

$$c1=delta+(1+r)*0.5*(S2*a2(-1)+S1*a1(-1))/S1-a1$$

  - this group gets members from groups 1 & 2

- First-order condition for group 1

$$a1 = 0;$$

Some Dynare equations

- Budget constraint for group 2, uuue, thus currently unemployed

  c2=delta+(1+r)*0.5*(S4*a4(-1)+S3*a3(-1))/S2-a2

  - this group gets members from groups 3 & 4

Some Dynare equations

- First-order condition for group 2

```
weight2*(c2-delta^(1+1/phi)/(1+1/phi))^-sigma
=
beta*(1+r(+1))*
(
0.5*weight9*(c9(+1)-le(+1)^(1+1/phi)/(1+1/phi))^-sigma
+
0.5*weight1*(c1(+1)-delta^(1+1/phi)/(1+1/phi))^-sigma
);
```

- Members of this group can go to group 1, *uuuu*, or group 9, *euuu*, with equal probability

Some Dynare equations

- Budget constraint for group 9, euuu, thus currently employed

  c9=w*le+(1+r)*0.5*(S2*a2(-1)+S1*a1(-1))/S9-a9

  - this group gets members from groups 1 & 2

Some Dynare equations

- First-order condition for group 9

```
weight9*(c9-le^(1+1/phi)/(1+1/phi))^-sigma
=
beta*(1+r(+1))*
(
0.8*weight13*(c13(+1)-le(+1)^(1+1/phi)/(1+1/phi))^-sigma
+
0.2*weight5*(c5(+1)-delta^(1+1/phi)/(1+1/phi))^-sigma
);
```

  - Members of this group can go to group 5, *ueuu*, or group 13, *eeuu*, with 0.2 and 0.8 probability, respectively

Modification: State dependent unemployment

State dependent Π

If $N = 2$, then one could have

$$S_t = \begin{bmatrix} .5 - \eta_{u\ u} A_{t-1} & .5 - \eta_{ue} A_{t-1} & 0 & 0 \\ 0 & 0 & .2 - \eta_{e\ u} A_{t-1} & .2 - \eta_{e\ e} A_{t-1} \\ .5 + \eta_{u\ u} A_{t-1} & .5 + \eta_{ue} A_{t-1} & 0 & 0 \\ 0 & 0 & .8 + \eta_{e\ u} A_{t-1} & .8 + \eta_{e\ e} A_{t-1} \end{bmatrix} S_{t-1}$$

Modification: State dependent unemployment

- Note that the columns sum up to 1
- Following LGR, dependence is on A_{t-1}, but could also be A_{t-1}
- !!! This works without complications *only if* the aggregate state still does not matter for which group is at the borrowing constraint

Other modifications

- Productivity of the employed could be different
  - for example, those who were recently unemployed are less productive

Appendix: Vermeylen approach

- Consider the following model

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \mathsf{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_t = \exp(\theta_t) k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

$$\theta_{t+1} = \begin{array}{c} \theta_L \text{ with probability } p(\theta|\theta_t) \\ \theta_H \text{ with probability } 1-p(\theta_t) \end{array}$$

- First-order perturbation:

$$k_t = \bar{k} + h_k(k_{t-1}-\bar{k}) + h_\theta(\theta_t - \bar{\theta})$$

- Thus, h_k is the same independent of the value of θ

First-order conditions

- policy function when $\theta_t = \theta_L$: $k_L(k_{t-1})$
- policy function when $\theta_t = \theta_H$: $k_H(k_{t-1})$
- Euler equation when $\theta_t = \theta_L$

$$(\theta_L k_{t-1}^{\alpha} - k_{L,t})^{-\gamma} = \begin{array}{c} p_{LL}\beta(\theta_L k_{L,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{L,t}^{\alpha-1} + 1 - \delta) \\ (1-p_{LL})\beta(\theta_H k_{L,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{L,t}^{\alpha-1} + 1 - \delta [\dots] \end{array}$$

- Euler equation when $\theta_t = \theta_H$

$$(\theta_H k_{t-1}^{\alpha} - k_{H,t})^{-\gamma} = \begin{array}{c} (1-p_{HH})\beta(\theta_L k_{H,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{H,t}^{\alpha-1} + 1 - [\dots] \\ p_{HH}\beta(\theta_H k_{H,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{H,t}^{\alpha-1} + 1 - \delta) \end{array}$$

- Auxiliary equation

$$k_{t-1} = (1 - z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}$$

- Now, θ_L & θ_H are fixed parameters and z_t is the stochastic variable.

New system with new variables

- Substitute out k_{t-1}. Now z_t enters the orginal Euler equations
- $k_{L,t}$ and $k_{H,t}$ have different steady state values
- Let the law of motion for z_t be given by

$$z_t - \bar{z} = \rho(z_{t-1})\left(z_{t-1} - \bar{z}\right) + \varepsilon_t. \tag{1}$$

$$\begin{aligned}
& \left(\theta_L\left((1-z_{t-1})k_{L,t-1}+z_{t-1}k_{H,t-1}\right)^{\alpha}-k_{L,t}\right)^{-\gamma} \\
= \quad & p_{LL}\beta(\theta_L k_{L,t}^{\alpha}-k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{L,t}^{\alpha-1}+1-\delta) \\
& (1-p_{LL})\beta(\theta_H k_{L,t}^{\alpha}-k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{L,t}^{\alpha-1}+1-\delta)
\end{aligned}$$

$$\begin{aligned}
& \left(\theta_H\left((1-z_{t-1})k_{L,t-1}+z_{t-1}k_{H,t-1}\right)^{\alpha}-k_{H,t}\right)^{-\gamma} \\
= \quad & (1-p_{HH})\beta(\theta_L k_{H,t}^{\alpha}-k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{H,t}^{\alpha-1}+1-\delta) \\
& p_{HH}\beta(\theta_H k_{H,t}^{\alpha}-k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{H,t}^{\alpha-1}+1-\delta)
\end{aligned}$$

$$z_t-\bar{z}=\rho(z_{t-1})\left(z_{t-1}-\bar{z}\right)+\varepsilon_t$$

- You could also use the linearized version of 1, since that is what will be using anyway

$$z_t = \bar{z} + \rho(\bar{z})(z_{t-1} - \bar{z}) + \varepsilon_t. \tag{2}$$

- The unconditional mean for z_t, $\bar{z} = \mathsf{E}[z_t]$, equals

$$\bar{z} = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}} \theta_H + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}} \theta_L$$

- The unconditional mean of $\rho(z_{t-1})$, $\rho(\bar{z})$ =$\mathsf{E}[\rho(z_t)]$, equals

$$\rho(\bar{z}) = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}}(2p_{HH} - 1) + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}}(2p_{LL} - 1)$$

Are new specification and original model consistent?

- In simulation use 1 not 2; so you have to do your own simulation
- We need
  - $z_t \in \{0, 1\}$
  - $\mathsf{E}[\varepsilon_t | z_{t-1} = 0] = E[\varepsilon_t | z_{t-1} = 1] = 0$
  - Conditional autocorrelations have to be correct
    - $\rho(1) = 2p_{HH} - 1$
    - $\rho(0) = 2p_{LL} - 1$

Are new specification and original model consistent?

To get that

$$z_t = z_{t-1} \text{ with prob } z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$$

set

$$\varepsilon_t = (1 - \rho(z_{t-1}))(z_{t-1} - \bar{z}) \text{ with prob } z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$$

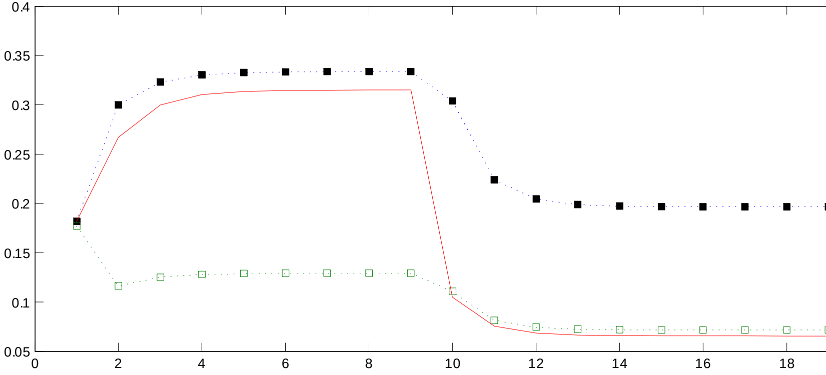
To get that

$$z_t = 1 - z_{t-1} \text{ with prob } z_{t-1}(1 - p_{HL}) + (1 - z_{t-1})(1 - p_{LL})$$

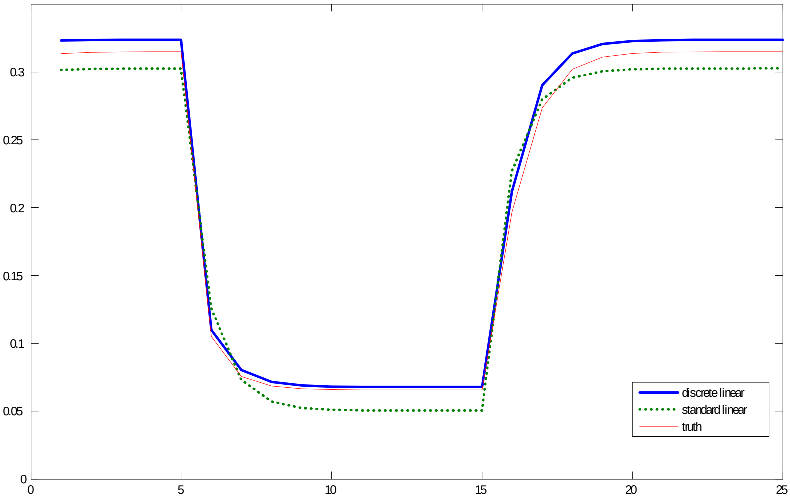
set

$$\varepsilon_t = -(1 + \rho(z_{t-1}))(z_{t-1} - \bar{z}) \text{ with prob } z_{t-1}(1 - p_{HH}) + (1 - z_{t-1})(1$$

log-linear discrete linearization



linear discrete versus standard linearization



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