Appendix to "Anticipated Growth and Business Cycles in Matching Models": Simplified Version with Analytical Results

Wouter J. DEN HAAN and Georg KALTENBRUNNER*

February 14, 2009

Abstract

Using a two-period version of the model, this appendix shows that Pigou cycles are not possible when the Hosios condition holds, that is, when the surplus is divided such that the competitive equilibrium corresponds to the social planners solution.

Overview. The matching friction used in this paper is similar to a quadratic adjustment cost; both imply that an increase in employment is too costly to implement within one period, but should be spread out over several periods. It is, thus, not surprising that investment in new projects increases in advance of the anticipated increase in productivity. To generate a Pigou cycle, however, it also must be true that the investment in new projects must—from society's point of view—be self-financing, that is, resources net of investments in new projects have to increase. If net resources do not increase, then it is not possible that each of the other spending components increases during the anticipation phase. In the main text, it is shown numerically that net resources do not increase in the version of

*Den Haan: Department of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands and CEPR, London, United Kingdom. Email: wdenhaan@uva.edu. Kaltenbrunner: McKinsey & Company, Inc., Magnusstraße 11, 50672 Köln, Germany. Email: Georg_Kaltenbrunner@mckinsey.com.

the model in which the entrepreneur's share satisfies the Hosios condition, but that they do increase when the entrepreneur's share is lower, that is, in the competitive equilibrium in which there is (from a social planner's point of view) underinvestment in new projects. This appendix illustrates this analytically using a simple two-period matching model. It is shown that if the Hosios condition is satisfied, that the model cannot generate Pigou cycles, but that this is possible when there is underinvestment in new projects.

Model. The economy consists of workers and entrepreneurs. At the beginning of each period, entrepreneurs decide how much to invest in new projects. The total amount invested in new projects is equal to $I_{N,t}$. The cost of starting a project is equal to one consumption commodity, so that $I_{N,t}$ is also equal to the total number of new projects started. A project is successful with probability λ_t. The total number of successful projects is assumed to be equal to $I_{N,t}^{\eta}$ with $0 < \eta < 1$. The probability of success, λ_t, is given by

$$\lambda_t = \frac{I_{N,t}^{\eta}}{I_{N,t}}, \tag{1}$$

which is decreasing in $I_{N,t}$. An unsuccessful project disappears and has no value. A successful project can start production in the same period. Production requires one worker and capital.

There are two periods. A successful investment in period 1 generates output in period 1 and continues in period 2 with probability $1-\delta$. Entrepreneurs can also start projects in period 2, but these are (if successful) only productive for one period. Workers have no disutility of labor and actual employment is determined by labor demand subject to the matching friction. Entrepreneurs and workers are part of a representative household that makes the consumption/savings decision.

The standard assumption in matching models is that successful matches become productive in the next period, whereas it is assumed here that a successful project becomes productive in the same period. The analysis here does not depend on the assumption made about the timing, but if the standard timing assumption would be adopted, then a three-period model would be needed. The idea about the two-period model is that the first period corresponds to the anticipation phase and the second period to the phase when the increase in productivity has been materialized.

Free-entry condition and firm problem. Entrepreneurs of successful projects maximize current-period profits. That is,

$$\max_{k_t} Z_t k_t^{\alpha} - R_t k_t,$$

where k_t stands for firm level capital, R_t for the rental rate, and Z_t for an exogenous productivity shock. The demand for capital is, thus, given by

$$\alpha Z_t k_t^{\alpha-1} = R_t, \quad t \in \{1,2\}. \tag{2}$$

In period 2, the free-entry condition is given by

$$1 = \lambda_2 \max_{k_2} \{Z_2 k_2^{\alpha} - R_2 k_2 - W_2\} \tag{3}$$

and in period 1 it is given by

$$\begin{aligned} 1 = \lambda_1 [\max_{k_1} \{Z_1 k_1^{\alpha} - R_1 k_1 - W_1\} \\ + \ \beta(1-\delta)\left(\frac{C_2}{C_1}\right)^{-\gamma} \max_{k_2} \{Z_2 k_2^{\alpha} - R_2 k_2 - W_2\}]. \end{aligned} \tag{4}$$

Here β is the discount factor and $\beta(C_2/C_1)^{-\gamma}$ the marginal rate of substitution defined using consumption of the representative household. The wage rate is assumed to be equal to

$$W_t = (1-\bar{\omega}_t)\left(Z_t k_t^{\alpha} - R_t k_t\right), \tag{5}$$

where $\bar{\omega}_t$ is exogenously given and determines the share of net revenues the entrepreneur receives in period t.

Using Equations (1), (2), and (5), the two free-entry conditions can be written as

$$1 = \bar{\omega}_2 I_{N,2}^{\eta-1} Z_2 (1-\alpha) k_2^{\alpha} \text{ and} \tag{6}$$

$$1 = \bar{\omega}_1 I_{N,1}^{\eta-1}\left[(1-\alpha) Z_1 k_1^{\alpha} + \beta(1-\delta)\left(\frac{C_2}{C_1}\right)^{-\gamma} \frac{\bar{\omega}_2}{\bar{\omega}_1}(1-\alpha) Z_2 k_2^{\alpha}\right]. \tag{7}$$

Employment and capital determination. The total number of productive projects is given by

$$N_1 = I_{N,1}^{\eta} \text{ and} \tag{8}$$

$$N_2 = (1-\delta)N_1 + I_{N,2}^{\eta}. \tag{9}$$

Besides the investment in new projects, there is also the investment in existing projects. This decision is made by the household who rents this type of capital out to existing firms. If the representative household would make both investment choices, then the model would correspond to a social planner's problem. The stock of capital in existing projects, K_t, is related to investment, I_t, according to

$$K_1 = I_1^{\eta} \text{ and} \tag{10}$$

$$K_2 = (1-\delta)K_1 + I_2^{\eta}. \tag{11}$$

In contrast, the amount invested in new projects, $I_{N,t}$, and associated vacancies are determined through a free-entry condition, which is a standard approach in labor market matching models. As discussed below, the outcome of the free-entry condition may not necessarily coincide with what the representative household or the social planner would choose.

The accumulation of capital is subject to diminishing returns, i.e., $0 < \eta < 1$, which mimics the matching friction in the creation of new projects. This assumption does not play an important role and is only made to make the two types of investment as symmetric as possible. For the same reason, the parameter δ controls both depreciation of capital and the destruction of successful projects. In other words, the only difference between the two types of investment is, thus, that one is determined by a free-entry condition and the other by a representative household.

Household problem. The representative household maximizes

$$\max_{C_1,C_2,I_1,I_2,K_1,K_2} \frac{C_1^{1-\gamma}}{1-\gamma} + \beta\frac{C_2^{1-\gamma}}{1-\gamma}$$

s.t.

$$
\begin{aligned}
C_1 + I_1 &= R_1 K_1 + P_1, && (12)\\
C_2 + I_2 &= R_2 K_2 + P_2, && (13)
\end{aligned}
$$

and the two accumulation equations (10) and (11). P_t is equal to the total amount of payments the household receives from entrepreneurs (profits minus investment in new projects) and workers. The household takes P_t as given. It is given by

$$
\begin{aligned}
P_t = N_t\left(Z_t k_t^{\alpha} - R_t k_t\right) - I_{N,t} &= N_t Z_t \left(\tfrac{K_t}{N_t}\right)^{\alpha} - R_t K_t - I_{N,t}\\
&= Z_t K_t^{\alpha} N_t^{1-\alpha} - R_t K_t - I_{N,t}.
\end{aligned} \tag{14}
$$

The first-order conditions for the household are given by

$$
1 = \eta I_2^{\eta-1} R_2 \text{ and} \tag{15}
$$

$$
1 = \eta I_1^{\eta-1}\left[R_1 + \beta(1-\delta)\left(\frac{C_2}{C_1}\right)^{-\gamma} R_2\right]. \tag{16}
$$

Using the equilibrium condition for the rental rate, these can be written as

$$
1 = \eta I_2^{\eta-1} \alpha Z_2 k_2^{\alpha-1} \text{ and} \tag{17}
$$

$$
1 = \eta I_1^{\eta-1}\left[\alpha Z_1 k_1^{\alpha-1} + \beta(1-\delta)\left(\frac{C_2}{C_1}\right)^{-\gamma} \alpha Z_2 k_2^{\alpha-1}\right]. \tag{18}
$$

Hosios condition and Pareto optimality. In the social planner's version of the model, the first-order conditions for investment in existing projects are identical to those of the competitive equilibrium, i.e., Equations (17) and (18). The social planner's first-order conditions for investments in new projects are equal to

$$
1 = \eta I_{N,2}^{\eta-1} Z_2 (1-\alpha) k_2^{\alpha} \text{ and} \tag{19}
$$

$$
1 = \eta I_{N,1}^{\eta-1}\left[(1-\alpha) Z_1 k_1^{\alpha} + \beta(1-\delta)\left(\frac{C_2}{C_1}\right)^{-\gamma} (1-\alpha) Z_2 k_2^{\alpha}\right]. \tag{20}
$$

These first-order conditions are identical to the free-entry conditions of the competitive equilibrium, i.e., Equations (6) and (7), *if* $\eta = \bar{\omega}_1 = \bar{\omega}_2$. This is the famous Hosios condition. The competitive equilibrium is not necessarily equal to the social planner's solution for two reasons. The first reason is that entrepreneurs are responsible for all costs of starting new projects, but only receive a share of the revenues. This leads to underinvestment in new projects. The second reason is that entrepreneurs take the probability of success as given and ignore that starting additional projects reduces the chance of success for other entrepreneurs. This leads to overinvestment in new projects. Under the Hosios condition these two effects exactly offset each other.

No Pigou cycles under optimal revenue sharing. The value of Z_1 is always set equal to 1, but the value of Z_2 can vary. The value of Z_2 is known in period 1, thus a change in Z_2 corresponds to an anticipated growth shock. A Pigou cycle exists if $dC_1/dZ_2 > 0$, $dI_{N,1}/dZ_2 > 0$, and $dI_1/dZ_2 > 0$; under these conditions the responses of employment and output are also positive.

Society's overall budget constraint in period 1 is given by

$$C_1 + I_{N,1} + I_1 = K_1^{\alpha} N_1^{1-\alpha} = Y_1. \tag{21}$$

A necessary condition for a Pigou cycle is that during the anticipation phase, i.e., period 1, resources net of investment in new projects, $Y_1 - I_{N,1}$, increase. If this does not happen, then it is impossible that both C_1 and I_1 increase during the anticipation phase. It is straightforward to show that no Pigou cycle exists if the Hosios condition holds. Differentiation of Equation (21) gives

$$\begin{aligned} dC_1 + dI_{N,1} + dI_1 &= (1-\alpha)\left(\frac{K_1}{N_1}\right)^{\alpha} dN_1 + \alpha \left(\frac{K_1}{N_1}\right)^{\alpha-1} dK_1 && (22) \\ &= (1-\alpha)\left(\frac{K_1}{N_1}\right)^{\alpha} \eta I_{N,1}^{\eta-1} dI_{N,1} + \alpha \left(\frac{K_1}{N_1}\right)^{\alpha-1} \eta I_1^{\eta-1} dI_1 && (23) \end{aligned}$$

or

$$dC_1 + \left[1 - (1-\alpha)\left(\frac{K_1}{N_1}\right)^{\alpha} \eta I_{N,1}^{\eta-1})\right] dI_{N,1} + \left[1 - \alpha \left(\frac{K_1}{N_1}\right)^{\alpha-1} \eta I_1^{\eta-1}\right] dI_1 = 0. \tag{24}$$

From the social planner's first-order conditions, it follows immediately that the two terms in square brackets are strictly positive. But the sum of three strictly positive elements cannot be equal to zero, so it is not possible that all three expenditure components increase.

Thus, although this model allows resources to increase in period 1, it never happens when the competitive equilibrium is Pareto optimal. For the discussion below, it is helpful to understand why the terms in square brackets are positive. Each term represents the cost of investing one more unit minus the contemporaneous increase in production. If these terms are negative, then the investment would already recover its cost in period 1. But an investment in period 1 leads to additional positive net revenues in period 2. This situation would, thus, not be optimal and corresponds to underinvestment from the social planner's point of view.

Pigou cycles when revenues are not shared optimally. Equation (24) is derived from the overall budget constraint and holds independent of whether the competitive equilibrium is Pareto optimal or not. For the model to generate a Pigou cycle, one of the two terms in square brackets must be negative. The second term in square brackets is related to investment in existing projects by the household. The household's first-order condition ensures that this term is always positive. The first term in square brackets is related to investment in new projects by entrepreneurs and this term is not necessarily positive. In particular, the term will be negative if $\bar{\omega}_1$ is sufficiently low. When $\bar{\omega}_1$ is low then entrepreneurs receive only a small share of the revenues, which implies that investment in new projects is low as well (and below the socially optimal level). If this effect is strong enough, then investments in new projects are so productive that (from a social planner's point of view) the costs are recovered in the first period.[1]

The analysis so far has shown that a low enough value of $\bar{\omega}_1$ ensures that $C_1 + I_1$ increases *if* $I_{N,1}$ increases. This is necessary for a full Pigou cycle, but not sufficient. It remains to be shown that $I_{N,1}$ indeed increases in the competitive equilibrium and that not only C_1 *plus* I_1, but both components increase. These two requirements will be discussed next.

$I_{N,1}$ robustly increases in response to an increase in Z_2 when wages increase less than proportionally with Z_2. The reason is that with (partially) sticky wages the increase in Z_2 leads to an expected increase in the entrepreneur's share of revenues making investment

[1] Because the entrepreneur only receives a share $\bar{\omega}_1$, this is not true from his point of view.

in new projects more attractive.

An increase in net resources induced by an increase in $I_{N,1}$ implies that C_1+I_1 increases. Consider the following two cases. First, if consumption smoothing is sufficiently important for the agent, i.e., the value of γ is high enough, then the reduction in the marginal rate of substitution dominates the increase in $R_2 + (1 - \delta)$, which implies that C_1 increases for sure, but I_1 possibly decreases.[2] Second, when $\gamma = 0$ then the household doesn't care about consumption smoothing and I_1 increases, but C_1 possibly decreases. $C_1 + I_1$ increases for both values of γ and for all values in between. The continuity of the problem implies that there is a value of γ such that both C_1 and I_1 increase.[3]

[2] The only caveat is that the value of γ should not be so high that the reduction in the marginal rate of substitution also leads to a reduction in $I_{N,1}$. With sticky wages the return on investment in new projects increases enough to prevent this from happening for reasonable values of γ. Also see Section **??**.

[3] For example, suppose that $\alpha = 0.5$, $\beta = 0.99$, $\gamma = 0.5$, $\eta = 0.5$, $\delta = 0.05$, $\bar{\omega}_1 = \bar{\omega}_2 = 0.1$, Z_2 increases with 1%, and $\bar{\omega}_2$ does not depend on Z_2, i.e., wages are proportional to profits. Then $I_{N,1}$ increases with 0.49%, I_1 increases with 0.33%, and C_1 increases with 0.06%. If $\bar{\omega}_2$ does increase from 0.1 to 0.11 when Z_2 increases, i.e., wages increase less than proportionally with wages, then $I_{N,1}$ increases with 3.3%, I_1 with 0.7%, and C_1 with 0.8%.