Nonlinear and Stable Perturbation-Based Approximations

Appendix

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Abstract

Appendix A.1 compares the differences between the alternative pruning procedures. Appendix A.2 shows that the n^{th}-order pruned perturbation approximation does not deliver an exact fit if the truth is a polynomial even though a pruned perturbation approximation is a polynomial. Appendix A.3 shows that our implementation of the pruning procedure is a proper perturbation approximation. Appendix B discusses details regarding the implementation of the perturbation-plus procedure. Appendix C shows that our numerical approximation obtained with a projection method is very accurate and can, thus, be used as a stand-in for the truth.

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A More on pruning

Kim, Kim, Schaumburg, and Sims (2008) and Lombardo (2010) describe how to construct a second-order pruned perturbation approximation. Lombardo (2010) also explains why his formulation of the second-order pruned perturbation approximation is a second-order local approximation. The formulation in Lombardo (2010) is somewhat different than the one used in Kim, Kim, Schaumburg, and Sims (2008) and is somewhat different than the one used in this paper.

This appendix has three objectives. First, we want to discuss the similarities and differences between the formulation used in Lombardo (2010) and the one used in this paper. Second, we want to explain why a second-order pruned perturbation approximation is a second-order approximation even though the second-order pruned perturbation approximation of a second-order polynomial is not simply that second-order polynomial. The third purpose of this appendix is to explain how to and how *not* to do higher-order pruning.

A.1 Different formulations for pruning

The differences are explained using a simple example. Throughout this subsection, we assume that the true law of motion is given by

$$\begin{aligned} x_t &= \rho_1 x_{t-1} + \rho_2 x_{t-1}^2 + \sigma\varepsilon_t, \quad (1) \\ \mathrm{E}\left[\varepsilon_t^2\right] &= 1. \quad (2) \end{aligned}$$

We assume that $|\rho_1| < 1$, which implies that the process is locally stable. The regular second-order perturbation approximation is simply equal to the function itself, that is,

$$x_t = \tilde{f}_{2^{\text{nd}}}(x_{-1}; \sigma) = \rho_1 x_{t-1} + \rho_2 x_{t-1}^2 + \sigma\varepsilon_t. \quad (3)$$

Lombardo (2010) generates the second-order pruned perturbation approximation using the following system of equations:

$$\tilde{x}_t^{(2)} = \sigma\tilde{x}_t^{[1]} + \sigma^2\tilde{x}_t^{[2]}, \tag{4a}$$

$$\tilde{x}_t^{[1]} = \rho_1\tilde{x}_{t-1}^{[1]} + \varepsilon_t, \tag{4b}$$

$$\tilde{x}_t^{[2]} = \rho_1\tilde{x}_{t-1}^{[2]} + \rho_2\left(\tilde{x}_{t-1}^{[1]}\right)^2. \tag{4c}$$

We generate the second-order pruned perturbation approximation using

$$\hat{x}_t^{(2)} = \rho_1\hat{x}_{t-1}^{(2)} + \rho_2\left(\hat{x}_{t-1}^{(1)}\right)^2 + \sigma\varepsilon_t, \tag{5a}$$

$$\hat{x}_t^{(1)} = \rho_1\hat{x}_{t-1}^{(1)} + \sigma\varepsilon_t. \tag{5b}$$

Suppose that

$$\tilde{x}_0^{(1)} = \tilde{x}_0^{(2)} = \hat{x}_0^{(1)} = \hat{x}_0 = x_0 = 0 \tag{6}$$

and

$$\varepsilon_t = 1 \text{ if } t = 1, \tag{7}$$

$$\varepsilon_t = 0 \text{ if } t > 1. \tag{8}$$

For this set of values for ε_t,[1]

$$\tilde{x}_t^{(2)} = \hat{x}_t^{(2)} \neq x_t \text{ and} \tag{9}$$

$$\tilde{x}_t^{(2)} - x_t = \hat{x}_t^{(2)} - x_t = O(\sigma^3). \tag{10}$$

Nevertheless, there is a difference between the two formulations. This is easy to see when $\sigma = 0$. The formulation according to equation (4) implies that

$$\tilde{x}_t^{(2)} = 0 \quad \forall t, \tag{11}$$

[1] A function $f(\sigma) = O(\sigma^m)$ if there exist an M such that

$$\lim_{\sigma\to 0}\frac{f(\sigma)}{\sigma^m} < M < \infty.$$

whereas the formulation according to equation (5) implies that

$$\hat{x}_t^{(2)} = \rho_1 \hat{x}_{t-1}^{(2)} + \rho_2 \left(\hat{x}_{t-1}^{(1)}\right)^2, \tag{12a}$$

$$\hat{x}_t^{(1)} = \rho_1 \hat{x}_{t-1}^{(1)}. \tag{12b}$$

That is, the formulation of Lombardo (2010) does not describe any transition dynamics, whereas our formulation does. Nevertheless, it is easy to show that both formulations are proper second-order approximations *if* an additional condition is satisfied. Suppose that x_t is generated by equation (1), $\tilde{x}_t^{(2)}$ is generated by equation (4), and $\hat{x}_t^{(2)}$ is generated by equation (5). Then it is easy to show that

$$\tilde{x}_t^{(2)} = x_t + O(\sigma^3) \text{ and} \tag{13}$$

$$\hat{x}_t^{(2)} = x_t + O(\sigma^3) \tag{14}$$

if

$$x_0 = \sigma^2 \bar{x}_0 \text{ with } \bar{x}_0 < \infty. \tag{15}$$

That is, our pruned perturbation formulation does allow transition dynamics, but the initial value chosen cannot be too far away from the steady state. Similarly, it is acceptable to ignore transition dynamics, but only if the initial value is close enough to the steady state. It is obvious that a condition like this is needed. There are values for x_0 such that the time path generated by the true law of motion given in equation (1) explodes, whereas the time paths generated by equations (4) and (5) never explode.

A.2 Pruned perturbation and inability to fit polynomials exactly

Pruned perturbation approximations as well as regular perturbation approximations are polynomials. If the truth is an n^{th}-order polynomial, then the *regular* n^{th}-order perturbation approximation would give an exact fit. But the pruned n^{th}-order perturbation approximation is not able to accomplish this. We use the following example to explain the reason.

Suppose that the truth is given by

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-1}^2 + \sigma \varepsilon_t \tag{16}$$

with

$$\mathrm{E}_t\left[\varepsilon_t^2\right] = 1.$$

The second-order pruned perturbation approximation is generated by the following system:[2]

$$x_t^{(1)} = \alpha_1 x_{t-1}^{(1)} + \sigma\varepsilon_t, \tag{17}$$

$$x_t^{(2)} = \alpha_1 x_{t-1}^{(2)} + \alpha_2\left(x_{t-1}^{(1)}\right)^2 + \sigma\varepsilon_t. \tag{18}$$

Although the law of motion for x_t, given in equation (18), closely resembles the true law of motion, given in equation (16), there is one fundamental difference. The difference is that the expression in equation (18) contains $x_{t-1}^{(1)}$, which is generated by a different law of motion than x_t.

Now consider higher-order approximations. When the truth is given by equation (16), then the third-order *regular* perturbation approximation is, of course, equal to the second-order regular perturbation approximation, which in turn is equal to the truth. But this is not true for the pruned perturbation approximation. The third-order pruned perturbation approximation is generated by the following set of equations:

$$x_t^{(1)} = \alpha_1 x_{t-1}^{(1)} + \sigma\varepsilon_t, \tag{19}$$

$$x_t^{(2)} = \alpha_1 x_{t-1}^{(2)} + \alpha_2\left(x_{t-1}^{(1)}\right)^2 + \sigma\varepsilon_t, \tag{20}$$

$$x_t^{(3)} = \alpha_1 x_{t-1}^{(3)} + \alpha_2\left(x_{t-1}^{(2)}\right)^2 + \sigma\varepsilon_t. \tag{21}$$

The third-order pruned perturbation approximation still does not match the true second-order polynomial, although the mistake has become smaller.[3]

There is a neat way to characterize the error made by the pruned perturbation approximation. Suppose one uses the pruned perturbation approximation to calculate the impulse response function (IRF) of a unit-shock to ε_t starting at the steady state. If the

[2]The inability to fit a regular polynomial is also true for the pruning approximation according to equation (4), $\tilde{x}_t^{(2)}$.

[3]The variable that is being squared is now being generated by a law of motion that is closer to the truth.

truth is a second-order polynomial, then the n^{th}-order pruned perturbation approximation will give the right values for the IRF for the first n periods. Thus, the second-order pruned perturbation approximation will give the right answer only up to the first two periods.

The question arises whether it is desirable that pruned perturbation approximations do not (approximately) replicate a polynomial when the truth is (approximately) a polynomial. Many functions in economic models can be approximated well with a low-order polynomial of the original state variables. The distortion that is introduced by the pruned perturbation approximations is in those cases likely to deteriorate the approximation. But not all functions are approximated well with a polynomial. That is likely to be the case when regular perturbation approximations explode and the true model solution does not. But the fact that regular polynomials provide a poor approximation does, of course, not imply that pruned perturbation approximations do. There are many aspects to a function and stability is only one of them.

A.3 Convergence of our pruning formulation

In this subsection, we discuss in more detail why our formulation of n^{th}-order pruned perturbation generates approximations that are of order $O(\sigma^{n+1})$.

A.3.1 Convergence of second-order pruning

To simplify the notation, we assume that the true law of motion is defined by

$$0 = \mathrm{E}_t\left[H(x_{t+1}, x_t, x_{t-1}, z_t; \sigma)\right], \tag{22}$$

$$z_t = \sigma\varepsilon_t, \tag{23}$$

where x_t and ε_t are scalars. The regular second-order perturbation approximation can be written as

$$X^{(2)}_{t;\sigma} = \begin{array}{l} \overline{f}'_x X^{(2)}_{t-1;\sigma} + \overline{f}'_z \sigma\varepsilon_t \\ +0.5\overline{f}''_{x^2}\left(X^{(2)}_{t-1;\sigma}\right)^2 + \overline{f}''_{xz} X^{(2)}_{t-1;\sigma}\sigma\varepsilon_t + 0.5\overline{f}''_{z^2}\left(\sigma\varepsilon_t\right)^2 \end{array}, \tag{24a}$$

$$x^{(2)}_{t;\sigma} = X^{(2)}_{t;\sigma} + \bar{x}_{2^{\text{nd}};\sigma}. \tag{24b}$$

Note that $X^{(2)}_{t;\sigma}$ is defined as the value x_t relative to the *stochastic* steady state of the second-order approximation, not relative to the non-stochastic steady state. This means that its value depends on σ. Our formulation of the pruned second-order perturbation approximation, $\widehat{X}^{(2)}_{t;\sigma}$ is given by

$$\begin{aligned}
\widehat{X}^{(1)}_{t;\sigma} &= \overline{f}'_x \widehat{X}^{(1)}_{t-1;\sigma} + \overline{f}'_z \sigma\varepsilon_t, && \text{(25a)}\\
\widehat{X}^{(2)}_{t;\sigma} &= \begin{array}{l} \overline{f}'_x \widehat{X}^{(2)}_{t-1;\sigma} + \overline{f}'_z \sigma\varepsilon_t \\ +0.5\overline{f}''_{x^2}\left(\widehat{X}^{(1)}_{t-1;\sigma}\right)^2 + \overline{f}''_{xz}\widehat{X}^{(1)}_{t-1;\sigma}\sigma\varepsilon_t + 0.5\overline{f}''_{z^2}\left(\sigma\varepsilon_t\right)^2 \end{array}, && \text{(25b)}\\
\widehat{x}^{(2)}_{t;\sigma} &= \widehat{X}^{(2)}_{t;\sigma} + \bar{x}_{2^{\text{nd}};\sigma}. && \text{(25c)}
\end{aligned}$$

The formulation for the pruned perturbation approximation used in the literature is given by

$$\begin{aligned}
\widetilde{X}^{(1)}_{t} &= \overline{f}'_x \widetilde{X}^{(1)}_{t-1} + \overline{f}'_z \sigma\varepsilon_t, && \text{(26a)}\\
\widetilde{X}^{(2)}_{t;\sigma} &= \bar{c}_{2^{\text{nd}};\sigma} + \begin{array}{l} \overline{f}'_x \widetilde{X}^{(2)}_{t-1} + \overline{f}'_z \sigma\varepsilon_t \\ +0.5\overline{f}''_{x^2}\left(\widetilde{X}^{(1)}_{t-1}\right)^2 + \overline{f}''_{xz}\widetilde{X}^{(1)}_{t-1}\sigma\varepsilon_t + 0.5\overline{f}''_{z^2}\left(\sigma\varepsilon_t\right)^2 \end{array}, && \text{(26b)}\\
\tilde{x}^{(2)}_{t;\sigma} &= \widetilde{X}^{(2)}_{t;\sigma} + \bar{x}, && \text{(26c)}
\end{aligned}$$

where $\bar{x}$ is the non-stochastic steady state.

Our formulation ensures that the stochastic steady state of $\widehat{X}^{(1)}_{t;\sigma}$ and $\widehat{X}^{(2)}_{t;\sigma}$ are both equal to zero, which ensures that the stochastic steady state of $\widehat{x}^{(2)}_{t;\sigma}$ is equal to the stochastic steady state of the original second-order perturbation approximation. In contrast, according to the formulation used in the literature the three variables, $X^{(2)}_{t;\sigma}$, $\widetilde{X}^{(1)}_{t;\sigma}$, and $\widetilde{X}^{(2)}_{t;\sigma}$ have three different steady state values. For the discussion in this section, this difference does not matter, because here we consider the case that $\sigma \longrightarrow 0$ and the three stochastic steady states would then converge to the non-stochastic steady state.[4]

The parameter σ plays two roles in the approximation. First, it controls the volatility of the driving process. Second, it affects the coefficients of the approximation. Here it

[4] As discussed in the main text, the motivation for our modification is the following. If higher-order perturbation introduces a correction for uncertainty, then it makes sense to apply this correction to *all* measures of the state variables that are introduced by the pruning procedure.

is convenient to separate these two roles. We let Ω indicate the value of volatility used to determine the coefficients of the perturbation approximation and we let σ indicate the value of the driving process. That is, we write the regular second-order perturbation approximation as

$$\begin{aligned} X^{(2)}_{t;\Omega} = \; & \overline{f}'_{x;\Omega} X^{(2)}_{t-1;\Omega} + \overline{f}'_{z;\Omega}\sigma\varepsilon_t \\ & +0.5\overline{f}''_{x^2;\Omega}\left(X^{(2)}_{t-1;\Omega}\right)^2 + 0.5\overline{f}''_{z^2;\Omega}\left(\sigma\varepsilon_t\right)^2 + \overline{f}''_{xz;\Omega} X^{(2)}_{t-1;\Omega}\sigma\varepsilon_t \end{aligned} \tag{27}$$

and the corresponding pruned perturbation approximation as

$$\begin{aligned} \widehat{X}^{(2)}_{t;\Omega} &= \overline{f}'_{x;\Omega}\widehat{X}^{(2)}_{t-1;\Omega} + \overline{f}'_{z;\Omega}\sigma\varepsilon_t \\ &\quad +0.5\overline{f}''_{x^2;\Omega}\left(\widehat{X}^{(1)}_{t-1;\Omega}\right)^2 + 0.5\overline{f}''_{z^2;\Omega}\left(\sigma\varepsilon_t\right)^2 + \overline{f}''_{xz;\Omega}\widehat{X}^{(1)}_{t-1;\Omega}\sigma\varepsilon_t \end{aligned} \tag{28a}$$

$$\widehat{X}^{(1)}_{t;\Omega} = \overline{f}'_{x;\Omega}\widehat{X}^{(1)}_{t-1;\Omega} + \overline{f}'_{z;\Omega}\sigma\varepsilon_t \tag{28b}$$

We know that the regular perturbation approximation given in equation (27) is a second-order approximation if σ and Ω approach zero.[5] Consequently, if the perturbation approximation given in equation (28) approaches the perturbation approximation given in equation (27) as $\sigma \longrightarrow 0$, then it is a second-order approximation of the truth if σ and Ω approach zero.

We will show that our pruned perturbation approximation converges to the regular perturbation approximation for fixed Ω, which implies that our procedure correctly approximates the correction that higher-order perturbation introduces for uncertainty (indicated by Ω) at least as $\sigma \longrightarrow 0$. For the regular pruning formulation this is only true if both $\Omega \longrightarrow 0$ and $\sigma \longrightarrow 0$.

The difference between $X^{(2)}_{t;\Omega}$ and $\widehat{X}^{(2)}_{t,\Omega}$ is equal to

$$\begin{aligned} X^{(2)}_{t;\Omega} - \widehat{X}^{(2)}_{t;\Omega} = \; & \overline{f}'_{x;\Omega}\left(X^{(2)}_{t-1;\Omega} - \widehat{X}^{(2)}_{t-1;\Omega}\right) \\ & +0.5\overline{f}''_{x^2;\Omega}\left(\left(X^{(2)}_{t-1;\Omega}\right)^2 - \left(\widehat{X}^{(1)}_{t-1;\Omega}\right)^2\right) \\ & +\overline{f}''_{xz;\Omega}\varepsilon_t\left(\sigma X^{(2)}_{t-1;\Omega} - \sigma\widehat{X}^{(1)}_{t-1;\Omega}\right) \end{aligned} \tag{29}$$

To see that $X^{(2)}_{t;\Omega} - \widehat{X}^{(2)}_{t;\Omega} = O(\sigma^3)$, first note that

$$X^{(2)}_{t-1;\Omega} - \widehat{X}^{(1)}_{t-1;\Omega} = O(\sigma^2), \tag{30}$$

[5] As in appendix A.1, one needs to assume that the initial condition converges to the steady state at the appropriate rate as σ approaches zero.

which means that

$$\sigma X^{(2)}_{t-1;\Omega} - \sigma \widehat{X}^{(1)}_{t-1;\Omega} = O(\sigma^3). \tag{31}$$

Moreover, since $X^{(1)}_{t-1;\Omega}$ is $O(\sigma^2)$ and $X^{(2)}_{t-1;\Omega}$ is $O(\sigma^3)$, it is also true that

$$\left(X^{(2)}_{t-1;\Omega}\right)^2 - \left(\widehat{X}^{(1)}_{t-1;\Omega}\right)^2 = O(\sigma^3). \tag{32}$$

Consequently, the only term remaining is $X^{(2)}_{t-1;\Omega} - \widehat{X}^{(2)}_{t-1;\Omega}$. But given the last two results this will be $O(\sigma^3)$ as long as the difference in the initial conditions is $O(\sigma^3)$ as in equation (15).

A.3.2 Convergence of higher-order pruning

The discussion above easily extends to the case for higher-order pruning. But one should be careful in specifying the formulation for higher-order pruned perturbation. We make this clear using a simple example.

Let the true law of motion be given by

$$\begin{aligned} x_t &= \rho_1 x_{t-1} + \rho_2 x^2_{t-1} + \rho_3 x^3_{t-1} + \sigma\varepsilon_t, && \text{(33a)} \\ \mathrm{E}\left[\varepsilon^2_t\right] &= 1. && \text{(33b)} \end{aligned}$$

As discussed in section 4.2 in the main text, the third-order pruned perturbation solution is given by

$$\begin{aligned} \hat{x}^{(3)}_t &= \rho_1 \hat{x}^{(3)}_{t-1} + \rho_2 \left(\hat{x}^{(2)}_{t-1}\right)^2 + \rho_3 \left(\hat{x}^{(1)}_{t-1}\right)^3 + \sigma\varepsilon_t && (34) \\ \hat{x}^{(2)}_t &= \rho_1 \hat{x}^{(2)}_{t-1} + \rho_2 \left(\hat{x}^{(1)}_{t-1}\right)^2 + \sigma\varepsilon_t && (35) \\ \hat{x}^{(1)}_t &= \rho_1 \hat{x}^{(1)}_t + \sigma\varepsilon_t && (36) \end{aligned}$$

Using the same logic as used in section A.3.1 for second-order pruning, it is easy to show that

$$x_t - \hat{x}^{(3)}_t = O(\sigma^4). \tag{37}$$

Now consider the following alternative formulation:

$$\begin{aligned} \tilde{x}^{(3)}_t &= \rho_1 \tilde{x}^{(3)}_{t-1} + \rho_2 \left(\tilde{x}^{(1)}_{t-1}\right)^2 + \rho_3 \left(\tilde{x}^{(1)}_{t-1}\right)^3 + \sigma\varepsilon_t && \text{(38a)} \\ \tilde{x}^{(1)}_t &= \rho_1 \tilde{x}^{(1)}_{t-1} + \sigma\varepsilon_t && \text{(38b)} \end{aligned}$$

This formulation does not generate a proper third-order approximation. To see that

$$x_t - \tilde{x}_t^{(3)} \neq O(\sigma^4) \tag{39}$$

consider the following example. Suppose that

$$x_0 = 0, \tag{40}$$

$$\varepsilon_t = 1 \text{ if } t = 1, \tag{41}$$

$$\varepsilon_t = 0 \text{ if } t > 1. \tag{42}$$

Table 1 reports the time paths for x_t and $\tilde{x}_t^{(3)}$. The term in bold, $2\rho_1\rho_2\sigma^3$, in the third period makes clear that the difference between x_t and $\tilde{x}_t^{(3)}$ is not of order $O(\sigma^4)$.

B Implementation of the perturbation-plus procedure

In this appendix, we describe a faster version of the perturbation-plus procedure and we describe in detail how we implemented the perturbation-plus procedure.

B.1 Faster implementation

An important factor that slows down the perturbation procedure is that there is no analytical solution to equation (30) in the main text, which we repeat here for convenience.

$$0 = \widetilde{\mathrm{E}}\left[H(\widetilde{f}_{1^{\mathrm{st}}}(x, z_{+1}), x, x_{-1}, z_{+1}, z)\right]. \tag{43}$$

The objective of the alternative formulation is to avoid using a nonlinear equation solver. Typically, x shows up more than once in $H(\cdot)$ and typically it is possible to find an analytical expression for x in terms of the other variables and x itself. If that is the case, then we can rewrite equation (43) as[6]

$$x = \widetilde{\mathrm{E}}\left[G(\widetilde{f}_{1^{\mathrm{st}}}(x, z_{+1}), x, x_{-1}, z_{+1}, z\right]. \tag{44}$$

[6] See section B.2 for an example.

Since this is still an equation in x, we have not made any progress. The idea is to use $\widetilde{f}_{1^{\text{st}}}(\cdot)$ not only for x_{+1} but also for the value of x inside $G(\cdot)$.[7] The one-step ahead modification is then defined as

$$x = \widehat{f}_{+1}(x_{-1}, z) = \widetilde{\mathrm{E}}_t\left[G(\widetilde{f}_{1^{\text{st}}}(\widetilde{f}_{1^{\text{st}}}(x_{-1}, z), z_{+1}), \widetilde{f}_{1^{\text{st}}}(x_{-1}, z), x_{-1}, z_{+1}, z\right]. \tag{45}$$

The two-step ahead modification is defined as

$$x = \widehat{f}_{+2}(x_{-1}, z) = \widetilde{\mathrm{E}}_t\left[G(\widehat{f}_{+1}(\widehat{f}_{+1}(x_{-1}, z), z_{+1}), \widehat{f}_{+1}(x_{-1}, z), x_{-1}, z_{+1}, z\right]. \tag{46}$$

Iteration on this scheme leads to the J-step ahead modification of the first-order perturbation solution.

B.2 Perturbation-plus and the neoclassical growth model

The first-order conditions for the neoclassical growth model are given by

$$c_t + k_t = e^{z_t} k_{t-1}^{\alpha} + (1-\delta)k_{t-1} \tag{47}$$

and

$$1 = \mathrm{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\alpha e^{z_{t+1}} k_t^{\alpha-1} + 1 - \delta\right)\right]. \tag{48}$$

The objective is to solve for k_t given values of k_{t-1} and z_t. We denote the solution as $k_t = \widehat{f}_{+1}(k_{t-1}, z_t)$. The solution for k_t and c_t are obtained using the following two equations

$$c_t + k_t = e^{z_t} k_{t-1}^{\alpha} + (1-\delta)k_{t-1} \tag{49a}$$

$$1 = \widetilde{\mathrm{E}}_t\left[\left(\frac{e^{z_{t+1}} k_t^{\alpha} + \left(1-\delta\right)k_t - \tilde{f}_{1^{\text{st}}}(k_t, z_{t+1})\right)}{c_t}\right)^{-\gamma}\left(\alpha e^{z_{t+1}} k_t^{\alpha-1} + 1 - \delta\right)\right]. \tag{49b}$$

where $\tilde{f}_{1^{\text{st}}}(k_t, z_{t+1})$ is the first-order perturbation solution for k_{t+1}. The conditional expectation $\widetilde{\mathrm{E}}_t[\cdot]$ is the numerical approximation to $\mathrm{E}_t[\cdot]$ using Gaussian-Hermite quadrature with five quadrature nodes.

[7] The difference between the original perturbation-plus procedure and its modification is very similar to the difference between time iteration and fixed-point iteration. See Chapter 17 in Judd (1998) for a discussion on these iteration schemes.

The procedure described in the last paragraph is the one-step ahead perturbation-plus procedure, because the behavior of next period's variables is described using the first-order perturbation procedure. The two-step ahead perturbation-plus approximation, $\widehat{f}_{+2}(\cdot)$, is the solution to equation (49) with $\tilde{f}_{1^{\text{st}}}(\cdot)$ replaced by $\widehat{f}_{+1}(\cdot)$.

B.3 Perturbation-plus and the matching model

The objective is to solve for λ_t given the values of the state variables, n_{t-1} and z_t. We denote this solution by

$$\lambda_t = \widehat{f}_{+1}(n_{t-1}, z_t). \tag{50}$$

We solve for λ_t from a system of five equations in five unknowns. The endogenous variables are λ_t, $p_{f,t}$, v_t, c_t, and n_t. The five equations are

$$\lambda_t = \beta \mathrm{E}\left[\left(\frac{\widetilde{c}_{t+1}(n_t, z_{t+1})}{c_t}\right)^{-\gamma}\left(\alpha e^{z_{t+1}} n_t^{\alpha-1} - w + (1-\rho_n)\widetilde{f}_{1^{\text{st}}}(n_t, z_{t+1})\right)\right], \tag{51}$$

and equations (7), (10), (12), and (14) from the main text, which we repeat here for convenience.

$$n_t = (1-\rho_n)\, n_{t-1} + p_{f,t} v_t, \tag{52}$$

$$\psi = p_{f,t}\, \lambda_t, \tag{53}$$

$$c_t = w n_{t-1} + (e^{z_t} n_{t-1} - w n_{t-1} - \psi v_t) = e^{z_t} n_{t-1} - \psi v_t, \tag{54}$$

$$p_{f,t} = \phi_0 \left(\frac{1-n_{t-1}}{v_t}\right)^{\phi}. \tag{55}$$

For this to be a system in five unknowns, we have to take a stand on how to determine $\widetilde{c}_{t+1}(n_t, z_{t+1})$. Since this is *next period's* consumption, we could use the first-order perturbation approximation. Instead we use for $\widetilde{c}_{t+1}(n_t, z_{t+1})$ the value that is implicitly defined by equations (52), (53), (54), and (55) from the main text for $t+1$ and

$$\lambda_{t+1} = \widetilde{f}_{1^{\text{st}}}(n_t, z_{t+1}). \tag{56}$$

Thus, we only use the first-order perturbation solution for λ_{t+1} and all the other variables are obtained using the true model equations.

The two-step ahead perturbation-plus approximation is defined analogously with $\tilde{f}_{1^{\text{st}}}(\cdot)$ replaced by $\widehat{f}_{+1}(\cdot)$.

Simplification. The functions $\widehat{f}_{+J}(\cdot)$ are only implicitly defined and its values are calculated using a nonlinear equation solver. Consequently, the cost of the algorithm increases sharply with J. In appendix B.1, we discussed a simplification that reduces the costs substantially. Here we discuss how this is implemented.

Instead of solving for λ_t, n_t and c_t simultaneously, we use the following procedure. First, calculate "temporary" values n_t and c_t using equations (52), (53), (54), and (55), and

$$\lambda_t = \widetilde{f}_{1^{\text{st}}}\left(n_{t-1}, z_t\right). \tag{57}$$

Denote the solutions for n_t and c_t as n_t^{temp} and c_t^{temp}. As above, the value for λ_t is calculated from

$$\lambda_t = \beta \mathrm{E}\left[\left(\frac{\widetilde{c}_{t+1}(n_t^{\text{temp}}, z_{t+1})}{c_t^{\text{temp}}}\right)^{-\gamma}\left(\begin{array}{c}\alpha e^{z_{t+1}}\left(n_t^{\text{temp}}\right)^{\alpha-1} - w \\ +(1-\rho_n)\widetilde{f}_{1^{\text{st}}}\left(n_t^{\text{temp}}, z_{t+1}\right)\end{array}\right)\right]. \tag{58}$$

With $\widetilde{c}_{t+1}(n_t^{\text{temp}}, z_{t+1})$ defined as above, we have an analytical expression for λ_t. The n_t^{temp} variable is only used to calculate λ_t. Given the solution for λ_t, the actual value for n_t is then obtained from equations (52), (53), (54), and (55), without making any further approximation. To calculate $\widehat{f}_{+J}(\cdot)$ one would use $\widehat{f}_{+J-1}(\cdot)$ instead of $\widetilde{f}_{1^{\text{st}}}(\cdot)$ both to calculate n_t^{temp}, c_t^{temp}, and to calculate λ_{t+1} in the Euler equation.

B.4 Perturbation-plus and the modified Deaton model

In each period of the simulation, i.e., given the value for cash on hand, x_t, we use an equation solver to calculate a_t from the Euler equation. We use Gaussian-Hermite quadrature to calculate the conditional expectation on the right-hand side of the Euler equation.[8]

For the one-step ahead modification, we use

$$\begin{aligned} c_{t+1} &= a_t + e^{z_{t+1}} - \frac{a_{t+1}}{1+r} && (59) \\ &\approx a_t + e^{z_{t+1}} - \frac{\widetilde{f}_{1^{\text{st}}}\left(a_t + e^{z_{t+1}}\right)}{1+r} && (60) \end{aligned}$$

[8] For the number of quadrature nodes we considered values between five and thirty and found that the results were robust to changing this number.

to calculate the realizations for consumption. This procedure defines the function $a_t = \widehat{f}_{+1}(x_t)$.

For the two-step ahead modification, we use $a_{t+1} = \widehat{f}_{+1}(x_{t+1})$ in the expression for consumption above. For each Gaussian-Hermite node, i.e., for each potential value of $a_t + e^{z_{t+1}}$, we use an equation solver to calculate a_{t+1} from tomorrow's Euler equation and on the right-hand side we use $a_{t+2} = \widetilde{f}_{1^{\text{st}}}(x_{t+2})$. Since $\widetilde{f}_{1^{\text{st}}}(\cdot)$ is only implicitly defined, we have to use an equation solver to solve for a_{t+1} for each quadrature node for z_{t+1}.

Although computational expensive, it is easy to iterate on this procedure to calculate the J-step ahead modification.

C Accuracy of our projection solutions

In this appendix, we document that the projection solutions that serve as a stand-in for the truth are very accurate.

C.1 Projection solution for matching model

We obtained a very accurate solution for the matching model using the following algorithm based on projection methods. We parameterized the policy function for the Lagrange multiplier λ by a linear spline that satisfies the Euler equation on each grid point. We have used $10,000$ equidistant grid points for n_{-1} ranging from 0.6 to 0.99. The other state variable, z, can take on two values, namely $-\zeta$ and $+\zeta$.

We used fixed-point iteration and the algorithm does the following at the i^{th} iteration. Starting point of the i^{th} iteration is the policy function from the last iteration, namely $\lambda = f^{(i-1)}(n_{-1}, z)$. Given this policy function it is straightforward to solve for the other variables. At grid point j, i.e., for given values of $n_{-1,(j)}$ and $z_{(j)}$, the value for λ is given by

$$\lambda_{(j)} = \beta \mathrm{E}\left[\left(\frac{c\left(n_{(j)}, z_{+1}\right)}{c_{(j)}}\right)^{-\gamma}\left(\alpha e^{z_{+1}} n_{(j)}^{\alpha-1} - w + (1-\rho_n)\lambda\left(n_{(j)}, z_{+1}\right)\right)\right]. \tag{61}$$

Integrating over the possible realizations for z_{+1} is trivial given that z has discrete support. If $\lambda\left(n_{(j)}, z_{+1}\right)$ and $c\left(n_{(j)}, z_{+1}\right)$ are determined using $f^{(i-1)}\left(n_{(j)}, z_{+1}\right)$, then one

can solve for $\lambda_{(j)}$, $n_{(j)}$, $v_{(j)}$, $p_{f,(j)}$, and $c_{(j)}$ by combining equation (61) with equations (52), (53), (54), and (55). This would be time iteration. To simplify the algorithm we use fixed-point iteration. The benefit of time iteration is that it has better convergence properties, but with the appropriate choice of the dampening parameter, the algorithm also converged with fixed-point iteration. We implemented fixed-point iteration as follows. First, calculate $\lambda_{(j)}^{\text{temp}} = f^{(i-1)}\left(n_{-1,(j)}, z_{(j)}\right)$. Use this value to calculate $n_{(j)}^{\text{temp}}$. Next, solve for λ using

$$\lambda_{(j)} = \beta \mathrm{E}\left[\left(\frac{c\left(n_{(j)}^{\text{temp}}, z_{+1}\right)}{c_{(j)}}\right)^{-\gamma}\left(\begin{array}{c}\alpha e^{z_{+1}}\left(n_{(j)}^{\text{temp}}\right)^{\alpha-1} - w \\ +(1-\rho_n)\lambda\left(n_{(j)}^{\text{temp}}, z_{+1}\right)\end{array}\right)\right], \tag{62}$$

where the values for c_{+1} and λ_{+1} are based on $f^{(i-1)}(\cdot)$. In principle one could set $f^{(i)}(n_{-1,(j)}, z_{(j)})$ equal to $\lambda_{(j)}$. But convergence may require a dampening factor, that is, to take a weighted average between $\lambda_{(j)}$ and $f^{(i-1)}(n_{-1,(j)}, z_{(j)})$. We iterate on this scheme until the maximum absolute difference is less than 1E−12.

As documented in table 1 in the main text, the errors made in the dynamic Euler equation accuracy test are minuscule.

C.2 Projection solution for modified Deaton model

We obtained an accurate solution for the modified Deaton model using projection techniques. The details are as follows. We parameterized the asset policy function by a linear spline that satisfies the Euler equation on each grid point. We use time iteration and we use the endogenous grid points algorithm of Carroll (2006). The advantage of time iteration (compared to fixed-point iteration) is that it has better convergence properties. The advantage of endogenous grid points is that there is an analytical solution for the variables when using time iteration. The disadvantage of using time iteration and endogenous grid points is that we have to specify a grid for both a_t and z_t, while strictly speaking there is only one state variable, namely cash on hand, $a_{t-1} + e^{z_t}$.

We used 1001 equidistant grid points for a_t ranging from -0.2 to 2 and we used 1001 equidistant grid points for z_t ranging from 0 to 3.

The projection algorithm is based on an iterative scheme that does the following at the i^{th} iteration. Starting point at the i^{th} iteration is the policy function from the last iteration, namely $a = f^{(i-1)}\left(a_{-1} + e^{z}\right)$. At grid point j, i.e., for given values of $a_{(j)}$ and $z_{(j)}$, the value for a_{-1} is solved from the Euler equation. The conditional expectation is approximated using Gaussian-Hermite quadrature with 30 nodes.[9] The possible realizations for c_{+1} are given by

$$c_{+1} = a_{(j)} + e^{z_{+1}} - \frac{f^{(i-1)}\left(a_{(j)} + e^{z_{+1}}\right)}{1+r}. \tag{63}$$

This leads to a set of combinations of a_{-1}, a, and z which gives $a = f(a_{-1} + e^{z})$. We iterate on this scheme until the maximum absolute difference between the values of $f^{(i-1)}$ and $f^{(i)}$ is less than 1E-7.

As documented in table 1 in the main text, the errors made in the dynamic Euler equation accuracy test are not quite as small as those for the matching model. This model is more difficult to solve, given that the variance is much higher. Nevertheless, the results are still good. The maximum error is 0.1% and the average error is 0.008%. These errors are minuscule relative to the errors of the perturbation based methods.

[9] One does not need that many nodes. But solving the model is not expensive so extra nodes do not hurt. In contrast, when using multi-step ahead perturbation the cost increases exponentially in the number of nodes. For that reason we used 5 nodes in the perturbation plus procedure, although we checked for robustness in some cases.

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Table 1: Dynamics for true and incorrect 3$^{\text{rd}}$-order pruned perturbation approximation

t	x_t	$\tilde{x}_t^{(3)}$
1	σ	σ
2	$\rho_1\sigma + \rho_2\sigma^2 + \rho_3\sigma^3$	$\rho_1\sigma + \rho_2\sigma^2 + \rho_3\sigma^3$
3	$\rho_1^2\sigma + \rho_1\rho_2\sigma^2 + \rho_1\rho_3\sigma^3$ $+\rho_2\left((\rho_1\sigma)^2 + \boldsymbol{2\rho_1\rho_2\sigma^3}\right)$ $+\rho_3(\rho_1\sigma)^3$ $+O(\sigma^4)$	$\rho_1^2\sigma + \rho_1\rho_2\sigma^2 + \rho_1\rho_3\sigma^3$ $+\rho_2(\rho_1\sigma)^2$ $+\rho_3(\rho_1\sigma)^3$

Notes: The values for x_t correspond to the values according to the true law of motion, which is given in equation (33a). The values for $\tilde{x}_t$ correspond to the values according to the incorrect formulation of the pruning procedure given in equation (38).