Solving Models with Heterogeneous Agents Reiter Method

Wouter J. Den Haan
London School of Economics

Background

Macroeconomic models with heterogeneous agents:

- idiosyncratic uncertainty large
  - $\Longrightarrow$ individual problem likely to be non-linear
  - $\Longrightarrow$ perturbation probably bad idea
- with idiosyncratic and *without* aggregate uncertainty
  - still doable even when problem is highly non-linear
- aggregate uncertainty small
  - $\Longrightarrow$ aggregate aspect of the problem probably easy
  - $\Longrightarrow$ perturbation likely to work

Crucial insight of Reiter (2008)

Combining these three facts suggest that we should

- combine perturbation and projection

Perturbation combined with Projection

- σ_e typically large $\Longrightarrow$ you can be far from the usual "$\sigma_z = 0, \boldsymbol{\sigma_e = 0}$" steady state
- Reiter's idea: Focus on the "$\sigma_z = 0$ and $\boldsymbol{\sigma_e > 0}$" steady state
- If $\sigma_z = 0 \Longrightarrow$ cross-sectional distribution doesn't change over time and the problem becomes much easier to solve
- Use perturbation to deal with $\sigma_z > 0$

Example environment for these slides

- Same as Krusell-Smith 1998 JPE paper **except** transition probabilities are assumed to constant
- In the appendix, it is discussed how to implement the Reiter method when transition probabilities do vary with the business cycle

Example environment

- Recall that KS assume that aggregate productivity, z_t, can take on only two values and transition probabilities vary such that employment can also take on only two values so that employment level is not needed as an additional state variable.
- When perturbation is used to deal with fluctuations in z_t then it is implicitly assumed that z_t can take on more than a finite number of values (even though one could restrict it to a finite number when simulating the model) $\Longrightarrow$ applying the Reiter method means that
  - either n_t becomes an additional state variable or
  - one should apply a modification of KS "trick" to get $z_t = n_t$ (see appendix)

Two key elements of Reiter procedure

1. A *numerical* solution to the model:

$$
\begin{aligned}
k_{i,t+1} &= P_N(e_{i,t}, k_{i,t}, z_t, m_t; \boldsymbol{\lambda_k}), \text{ or} \\
k_{i,t+1} &= 0 \text{ if } P_N(e_{i,t}, k_{i,t}, z_t, m_t; \boldsymbol{\lambda_k}) < 0
\end{aligned}
$$

   m_t is a characterization of the distribution

2. Knowing λ_k should be enough to write down a formula for $\Gamma_{\lambda_k}(\cdot)$, where

$$
m_{t+1} = \Gamma_{\lambda_k}(z_{t+1}, z_t, m_t)
$$

   "formula" means an exact algebraic expression (think: something that can be entered in the Dynare model block)

What does the second element require?

- An exact expression (i.e., formula) is required $\Longrightarrow \Gamma_{\lambda_k}$ cannot be such that it has to be determined with a simulation method or a subroutine
- Possibilities:

  ❶ m_t describes complete distribution $\Longrightarrow$

  - m_t can be histogram values at a fine grid (as in Reiter 2008). Simulation slides give expression for $\Gamma_{\lambda_k}(\cdot)$
  - m_t could be limited set of moments *if* it is combined with a distributional assumption as in Winberry (2016)

  ❷ m_t are the moments of the *levels* of $k_{i,t}$ so that explicit aggregation is possible as in XPA
  (this will introduce additional policy functions if higher-order moments are used. See XPA slides)

Rewrite the policy function

- Rewrite the *numerical* solution to the model as

$$\begin{aligned} k_{i,t+1} &= P_n(e_{i,t}, k_{i,t}; \boldsymbol{\lambda_{k,t}}) \text{ or} \\ k_{i,t+1} &= 0 \text{ if } P_n(e_{i,t}, k_{i,t}; \boldsymbol{\lambda_{k,t}}) < 0 \end{aligned}$$

with

$$\boldsymbol{\lambda}_{k,\boldsymbol{t}} = \lambda_k(z_t, m_t) = \lambda_k(s_t)$$

More on distribution

- Take KS environment. $e_{i,t} \in \{0,1\}$ and $z_t \in \left\{z^b, z^g\right\}$ **but** transition probabilities constant

- Suppose m_t contains the mean and uncentered variance of capital holdings for employed and unemployed **and** we make an assumption on the functional form of the distribution

- Thus, we know the density $f\left(k_{i,t}; m_{[1],0,t}, m_{[2],0,t}\right)$ for unemployed and density $f\left(k_{i,t}; m_{[1],1,t}, m_{[2],1,t}\right)$ for employed

- Notation: $m_{[k],e,t}$ is the k-order moment for capital of workers with employment status e in period t

More on distribution

- Expressions for end-of-period moments are easy to write down. E.g., for the second moment for the unemployed we get

$$\widehat{m}_{[2],0,t} = \int_{-\infty}^{+\infty} \left(P_N(0, k_{i,t}; \boldsymbol{\lambda_k})\right)^2 f\left(k_{i,t}; m_{[1],0,t}, m_{[2],0,t}\right) dk_{i,t}$$

- We use quadrature to turn this into a formula we can write down in say the Dynare model block ($\int$ becomes a sum)

- Getting a formula for beginning-of-next-period moments (which takes into account change in employment status), is just accounting (see simulation slides)

Notation & grid

- ε_j and κ_j: employment status and capital at grid point j
- Dimension of $\boldsymbol{\lambda}_{k,\boldsymbol{t}} = n^{\#}_{\lambda_k}$
  - If $P_N(\cdot)$ is 2nd-order complete polynomial $\Longrightarrow n^{\#}_{\lambda_k} = 6$
  - number of grid points $= n^{\#}_{\text{grid}} \geq n^{\#}_{\lambda_k}$
- no grid for s in the Reiter method !!!!!!!!!!!!!!!!

Model equation at grid points

- log utility and $\delta = 1$ for simplicity $\Longrightarrow$ Euler equation becomes

$$\left(r(s)\kappa_j + w(s)\varepsilon_j\bar{l} - P_N(\varepsilon_j, \kappa_j; \lambda_k(s))\right)^{-1}$$

$$=$$

$$\mathbb{E}\left[\beta r(s') \left(\begin{array}{c} (r(s'))\, P_N(\varepsilon_j, \kappa_j; \lambda_k(s)) + w(s)\varepsilon'\bar{l} \\ -P_N(\varepsilon', P_N(\varepsilon_j, \kappa_j; \lambda_k(s)); \lambda_k(s')) \end{array} \right)^{-1} \middle| \varepsilon_j, s \right]$$

  if $P_N(\varepsilon_j, \kappa_j; \lambda_k(s)) > 0$.

- This equation is replaced by $k_{i,t+1} = 0$ if $P_N(\varepsilon_j, \kappa_j; \lambda_k(s)) \leq 0$.
- !!!! One must make a guess which part of the two-part Kuhn-Tucker conditions should be used at each grid point. (With perturbation we consider small changes in z so it is reasonable to assume that these characterizations will not change with z_t)

Additional exact expressions

1. $r(s) = \alpha z \left(K/\bar{l}\right)^{\alpha-1}$
2. $w(s) = (1-\alpha) z \left(K/\bar{l}\right)^{\alpha}$
3. law of motion for z' and ε'
4. $m' = \Gamma_{\lambda_k}(z', z, m)$
   - m is histogram in Reiter $\Longrightarrow \Gamma_{\lambda_k}$ is fully known (see simulation slides)
   - m can also be a limited set of moments *if* it is combined with a functional form assumption as in example above

Mental break

- Have I really done anything?
- Not much
    - I constructed a grid
    - I construct a system with individual choices substituted out using $P_n(e_{i,t}, k_{i,t}; \boldsymbol{\lambda}(s_t))$

Perturbation system

- Suppose 2nd-order polynomial is used: $n^{\#}_{\lambda_k} = 6$
- Suppose there are 6 grid points
- After substituting out r and w as well as taking care of $\mathbb{E}_t\left[\cdot\right]$, we get the following type of system

$$\begin{array}{ccc} \underset{6\times 1}{F(\lambda_k\left(s\right),s)} & = & \underset{6\times 1}{0} \\ z' & = & \left(1-\rho\right)\bar{z}+\rho z+\varepsilon_z \\ m' & = & \Gamma_{\lambda_k}(z',z,m) \end{array}$$

- Thus,
  - $F(\cdot)$ known
  - $\lambda_k\left(s\right)$ unknown
- This is a standard perturbation system!!!!!!!!!

What is known and unknown?

- We can replace $\mathbb{E}\left[\cdot|\varepsilon_j, s\right]$ with a formula, either because variables have discrete support as in the KS environment or because we use quadrature approximaton
- The unknown in this system is $\lambda_k(s)$

Small comment

- If number of grid points exceeds $n^{\#}_{\lambda_k}$, then you have to take a stand on how to weigh the elements of $F(\cdot)$ to get a system of $n^{\#}_{\lambda_k}$ equations.

Perturbation system

- What are state variables **in this perturbation system**?

Perturbation system

- What are state variables **in this perturbation system**?
    - state variables: s

Perturbation system

- What are state variables **in this perturbation system**?
  - state variables: s
- What are *not* state variables **in this perturbation system**?

Perturbation system

- What are state variables **in this perturbation system**?
  - state variables: s
- What are *not* state variables **in this perturbation system**?
  - not state variables: ε and κ

Perturbation system

- What are state variables **in this perturbation system**?
  - state variables: s
- What are *not* state variables **in this perturbation system**?
  - not state variables: ε and κ
- What is this perturbation system solving for?

Perturbation system

- What are state variables **in this perturbation system**?
  - state variables: s
- What are *not* state variables **in this perturbation system**?
  - not state variables: ε and κ
- What is this perturbation system solving for?
  - It will give you policy functions for the elements of $\lambda_k(s)$. **These describe how the coefficients of the individual policy rules fluctuate with s**

A simple perturbation system?

- Reiter (2008) uses a fine histogram to characterize CDF
  - $\implies$ dimension of m typically high ($> 1,000$ in Reiter (2008))
  - $\implies \lambda_k$ has many inputs $\implies$ the perturbation system has to solve for many policy functions; the perturbation system solves how *each* element of λ_k changes with *each* element of s_t!!!!!!
  - $\implies$ higher-order perturbation becomes very tough (even first-order may be tricky)
- Winberry (2016) approach using moments makes problem more tractable

Steady state

- What is the steady state of this system?

Steady state

- What is the steady state of this system?
- Simply set $z' = z = \bar{z}$ and $m' = m = \overline{m}$

Steady state

- What is the steady state of this system?
- Simply set $z' = z = \overline{z}$ and $m' = m = \overline{m}$
- Then this system of $n^{\#}_{\lambda_k}$ equations solves the "no aggregate uncertainty" version of the model
(Intuitively: given $\overline{m}$ (and implied prices) the six equations associated with the capital Euler equation at the six grid points solve for the individual policy function. Given the policy function, one can solve for the steady-state distribution $\overline{m}$ (see the simulation slides that this is actually very easy if m is a histogram)

More general environment

- Above we assumed transition probabilities are fixed $\Longrightarrow$ (un) employment rate is constant
- Alternatives
  1. Specify how transition probabilities vary with z_t. This does not complicate implementation of the Reiter method except that in general it will introduce employment as an additional aggregate state variable
  2. Adopt a trick similar to the one used by KS that works for more general fluctuations in z_t. Specifically,
     - Transition probability from employed to employed $= \overline{p} z_{t+1}/z_t$
     - Transition probability from unemployed employed $=$ $(1-\overline{p})z_{t+1}/(\widetilde{n}-z_t)$, where $\widetilde{n}$ equals total work force. This probability is higher when z_t is higher (which can be justified by having less congestion on the matching market)
     - The law of motion for n_{t+1} is then given by

$$n_{t+1} = \overline{p}\frac{z_{t+1}}{z_t} z_t + (1-\overline{p})\frac{z_{t+1}}{(\widetilde{n}-z_t)}(\widetilde{n}-z_t) = z_{t+1}.$$

References

- Reiter, M., 2009, Solving heterogeneous-agent models by projection and perturbation, *Journal of Economic Dynamics and Control*, 32, 1120-1155.
- Winberry, T. 2016, A toolbox for solving and estimating heterogeneous agent macro models. Available at http://faculty.chicagobooth.edu/thomas.winberry/research/winber