Solving Models with Heterogeneous Agents - KS algorithm

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Individual agent

- Subject to employment shocks ($\varepsilon_{i,t} \in \{0, 1\}$)
- Incomplete markets
  - only way to save is through holding capital
  - borrowing constraint $k_{i,t+1} \geq 0$

Laws of motion

- z_t can take on two values
- $\varepsilon_{i,t}$ can take on two values
- probability of being (un)employed depends on z_t
- transition probabilities are such that unemployment rate only depends on current z_t. Thus:
  - $u_t = u^b$ if $z_t = z^b$
  - $u_t = u^g$ if $z_t = z^g$
  - with $u^b > u^g$.

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1-\tau_t) w_t \bar{l} \varepsilon_{i,t} + \mu w_t (1-\varepsilon_{i,t}) + (1-\delta) k_{i,t}$$
$$k_{i,t+1} \geq 0$$

for **given** processes of r_t and w_t, this is a relatively simple problem

Firm problem

$$
\begin{aligned}
r_t &= z_t \alpha \left(\frac{K_t}{\bar{l}(1 - u(z_t))} \right)^{\alpha - 1} \\
w_t &= z_t (1 - \alpha) \left(\frac{K_t}{\bar{l}(1 - u(z_t))} \right)^{\alpha}
\end{aligned}
$$

Government

$$\tau_t w_t \bar{l}(1 - u(z_t)) = \mu w_t u(z_t)$$

$$\tau_t = \frac{\mu u(z_t)}{\bar{l}(1 - u(z_t))}$$

What aggregate variables do agents care about?

- r_t and w_t
- They only depend on aggregate capital stock and z_t
- !!! This is not true in general for equilibrium prices
- Agents are interested in all information that forecasts K_t
- In principle that is the complete cross-sectional distribution of employment status and capital levels

Equilibrium - first part

- Individual policy functions solving agent's max problem
- A wage and a rental rate given by equations above.

Equilibrium - second part

- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

  - f_t = beginning-of-period cross-sectional distribution of capital and the employment status after the employment status has been realized.
  - z_{t+1} does not affect the cross-sectional distribution of capital but does affect the joint cross-sectional distribution of capital and employment status

Key approximating step

1. Approximate cross-sectional distribution with limited set of "characteristics"
   - Proposed in Den Haan (1996), Krusell & Smith (1997,1998), Rios-Rull (1997)
2. Solve for aggregate policy rule
3. Solve individual policy rule for a given aggregate law of motion
4. Make the two consistent

Krusell-Smith (1997,1998) algorithm

- Assume the following approximating aggregate law of motion

$$m_{t+1} = \bar{\Gamma}(z_{t+1}, z_t, m_t; \eta_{\bar{\Gamma}}).$$

- Start with an initial guess for its coefficients, $\eta^0_{\bar{\Gamma}}$

Krusell-Smith (1997,1998) algorithm

- Use following iteration until $\eta_{\bar{\Gamma}}^{iter}$ has converged:
  - Given $\eta_{\bar{\Gamma}}^{iter}$ solve for the individual policy rule
  - Given individual policy rule simulate economy and generate a time series for m_t
- Use a regression analysis to update values of η

$$\eta_{\bar{\Gamma}}^{iter+1} = \lambda \hat{\eta}_{\bar{\Gamma}} + (1-\lambda)\eta_{\bar{\Gamma}}^{iter}, \text{ with } 0 < \lambda \leq 1$$

Solving for individual policy rules

- Given aggregate law of motion $\Longrightarrow$ you can solve for individual policy rules with your favourite algorithm

- But number of state variables has increased:
  - State variables for agent: $s_{i,t} = \{\varepsilon_{i,t}, k_{i,t}, s_t\}$
  - with $s_t = \{z_t, m_t\} = \{z_t, K_t, \tilde{m}_t\}$.

Solving for individual policy rules

- s_t must "reveal" K_t
  - $s_t \Longrightarrow K_t \Longrightarrow r_t$ and r_t
- Let $K_{t+1} = \bar{\Gamma}_K(z_{t+1}, z_t, s_t; \eta_{\bar{\Gamma}})$, $\tilde{m}_{t+1} = \bar{\Gamma}_{\tilde{m}}(z_{t+1}, z_t, s_t; \eta_{\bar{\Gamma}})$
- If s_t includes many characteristics of the cross-sectional distribution $\Longrightarrow$ high dimensional individual policy rule

Individual policy rules & projection methods

First choice to make:

- Which function to approximate?
- Here we approximate $k_i(\cdot)$

$$k_{i,t+1} = P_n(s_{i,t}; \eta_{P_n})$$

- N_η : dimension η_{P_n}

Individual policy rules & projection methods

Next: Design grid

- s_κ the κ^{th} grid point
- $\{s_\kappa\}_{\kappa=1}^{\chi}$ the set with χ nodes
- $s_\kappa = \{\varepsilon_\kappa, k_\kappa, s_\kappa\}$, and $s_\kappa = \{z_\kappa, K_\kappa, \tilde{m}_\kappa\}$

Individual policy rules & projection methods

Next: Implement projection idea

1. Substitute approximation into model equations until you get equations of only
    1. current-period state variables
    2. coefficients of approximation, η_{P_n}
2. Evaluate at χ grid points $\Longrightarrow \chi$ equations to find η_{P_n}
    - $\chi = N_\eta \Longrightarrow$ use equation solver
    - $\chi > N_\eta \Longrightarrow$ use minimization routine

Individual policy rules & projection methods

First-order condition

$$
\begin{aligned}
c_t^{-\nu} &= \mathsf{E}\left[\begin{array}{c}\beta(r(z',K')+(1-\delta))\times \\ c_{t+1}^{-\nu}\end{array}\right] \\
\left(\text{income}_{i,t}-k_{i,t+1}\right)^{-\nu} &= \mathsf{E}\left[\begin{array}{c}\beta(r(z',K')+(1-\delta))\times \\ \left(\text{income}_{i,t+1}-k_{i,t+2}\right)^{-\nu}\end{array}\right]
\end{aligned}
$$

Individual policy rules & projection methods

First-order condition

$$\left(\begin{array}{c} (r(z_\kappa, K_\kappa) + 1 - \delta)k_\kappa \\ +(1-\tau(z_\kappa))w(z_\kappa, K_\kappa)\bar{l}\varepsilon_\kappa + \mu w(z_\kappa, K_\kappa)(1-\varepsilon_\kappa) \\ -P_n(s_\kappa; \eta_{P_n}) \end{array} \right)^{-\nu}$$

$$= \mathsf{E}\left[\begin{array}{c} \beta(r(z', K') + (1-\delta))\times \\ \left(\begin{array}{c} (r(z', K') + 1 - \delta)P_n(s_\kappa; \eta_{P_n}) \\ +(1-\tau(z'))w(z', K')\bar{l}\varepsilon' + \mu w(z', K')(1-\varepsilon') \\ -P_n(s'; \eta_{P_n}) \end{array} \right)^{-\nu} \end{array} \right]$$

Individual policy rules &projection methods

Euler equation errors:

$$u_\kappa = \begin{pmatrix} (r(z_\kappa, K_\kappa) + 1 - \delta)k_\kappa \\ +(1-\tau(z_\kappa))w(z_\kappa, K_\kappa)\bar{l}\varepsilon_\kappa + \mu w(z_\kappa, K_\kappa)(1-\varepsilon_\kappa) \\ -P_n(s_\kappa; \eta_{P_n}) \end{pmatrix}^{-\nu} -$$

$$\sum_{z' \in \{z^b, z^g\}} \sum_{\varepsilon' \in \{0,1\}} \begin{bmatrix} \beta(r(z', K') + (1-\delta)) \times \\ \begin{pmatrix} (r(z', K') + 1 - \delta)P_n(s_\kappa; \eta_{P_n}) \\ +(1-\tau(z'))w(z', K')\bar{l}\varepsilon' \\ +\mu w(z', K')(1-\varepsilon') \\ -P_n(\boldsymbol{s'}; \eta_{P_n}) \end{pmatrix}^{-\nu} \times \\ \pi(\varepsilon', z' \mid z_\kappa, \varepsilon_\kappa) \end{bmatrix}$$

Error depends on known variables and η_{P_n} when using

$$
\begin{aligned}
r(z_\kappa, K_\kappa) &= \alpha z_\kappa (K_\kappa / L(z_\kappa))^{\alpha-1} \\
w(z_\kappa, K_\kappa) &= (1-\alpha) z_\kappa (K_\kappa / L(z_\kappa))^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
r(z', K') &= \alpha z' (K' / L(z'))^{\alpha-1} \\
&= \alpha z' (\bar{\Gamma}_K(z', z_\kappa, s_\kappa; \eta_{\bar{\Gamma}}) / L(z'))^{\alpha-1} \\
w(z', K') &= (1-\alpha) z' (K' / L(z'))^{\alpha} \\
&= (1-\alpha) z' (\bar{\Gamma}_K(z', z_\kappa, s_\kappa; \eta_{\bar{\Gamma}}) / L(z'))^{\alpha}
\end{aligned}
$$

$$
\tau(z) = \mu(1 - L(z)) / \bar{l} L(z))
$$

$$
s' = \left\{ \begin{array}{c} k', \varepsilon', z', \\ \bar{\Gamma}_K(z', z_\kappa, s_\kappa; \eta_{\bar{\Gamma}}), \bar{\Gamma}_{\tilde{m}}(z', z_\kappa, s_\kappa; \eta_{\bar{\Gamma}}) \end{array} \right\}
$$

Again standard projection problem

- Find η_{P_n} by minimizing $\sum_{\kappa=1}^{\chi} u_{\kappa}^2$

Remaining issues

1. Using just the mean and approximate aggregation
2. Simulation
3. Other models and always ensuring equilibrium

Approximate aggregation

- The mean is often sufficient $\not\Rightarrow$ close to complete markets
- Why does only the mean matter?

Approximate aggregation

- Approximate aggregation $\equiv$
  - Next period's prices can be described quite well using
    - exogenous driving processes
    - means of current-period distribution
- Approximate aggregation
  - $\neq$ aggregates behave as in RA economy
    - with *same* preferences
    - with any preferences
  - $\neq$ individual consumption behaves as aggregate consumption

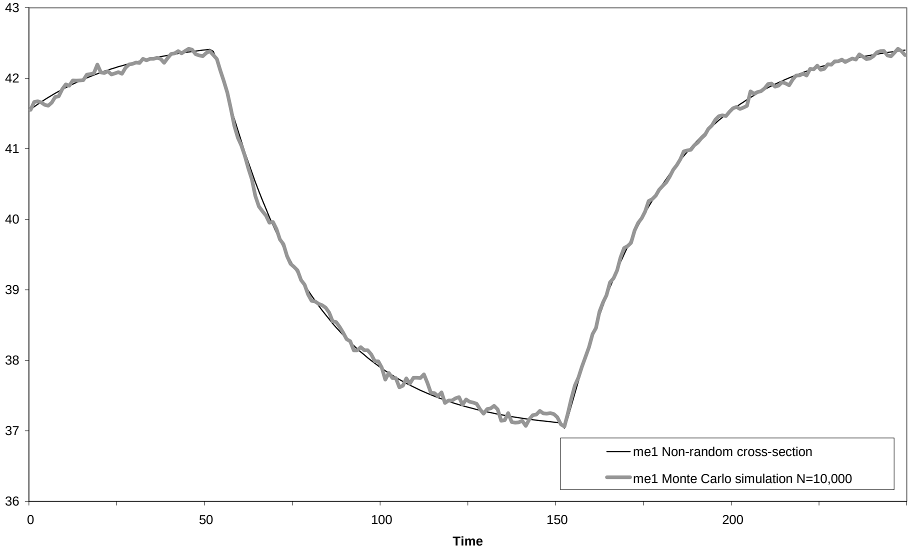
Why approximate aggregation

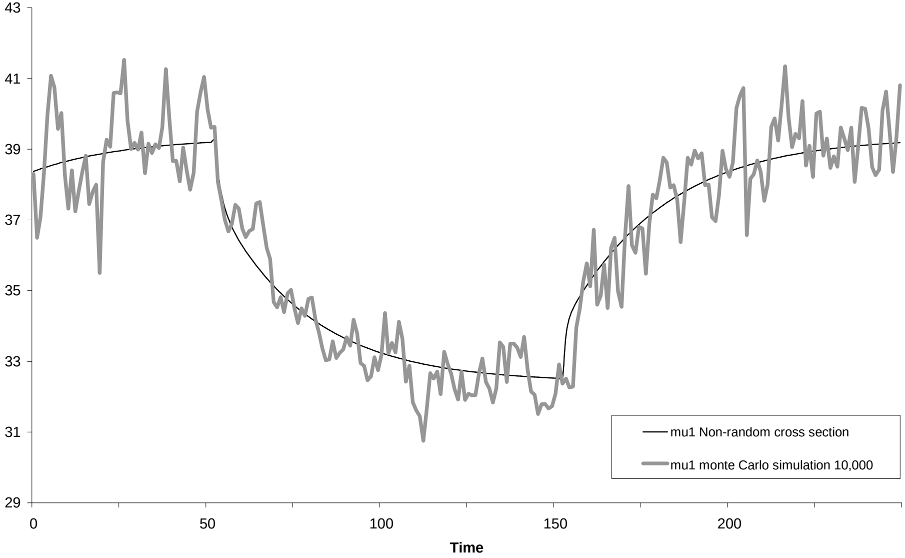
- If policy function *exactly* linear in levels
  - so also not loglinear
- then redistributions of wealth don't matter at all$\Longrightarrow$
  Only mean needed for calculating next period's mean
- Approximate aggregation still possible with non-linear policy functions
  - but policy functions must be sufficiently linear where it matters

How to simulate?

- Numerical procedure with a continuum of agents
- What if you really do like to simulate a panel with a finite number of agents?
  - Impose truth as much as possible: if you have 10,000 agents have exactly 400 (1,000) agents unemployed in a boom (recession)
  - Even then sampling noise is non-trivial

Simulation and sampling noise





43
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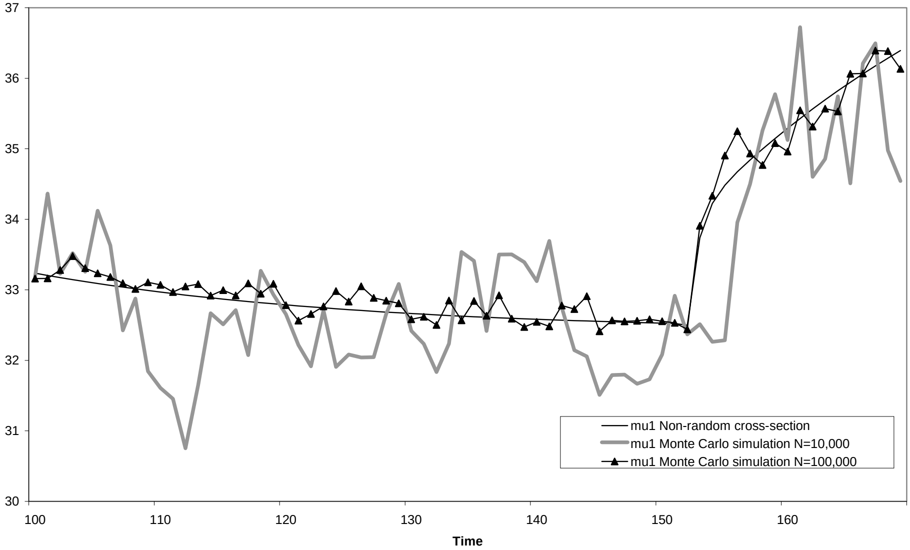
Time
mu1 Non-random cross section
mu1 monte Carlo simulation 10,000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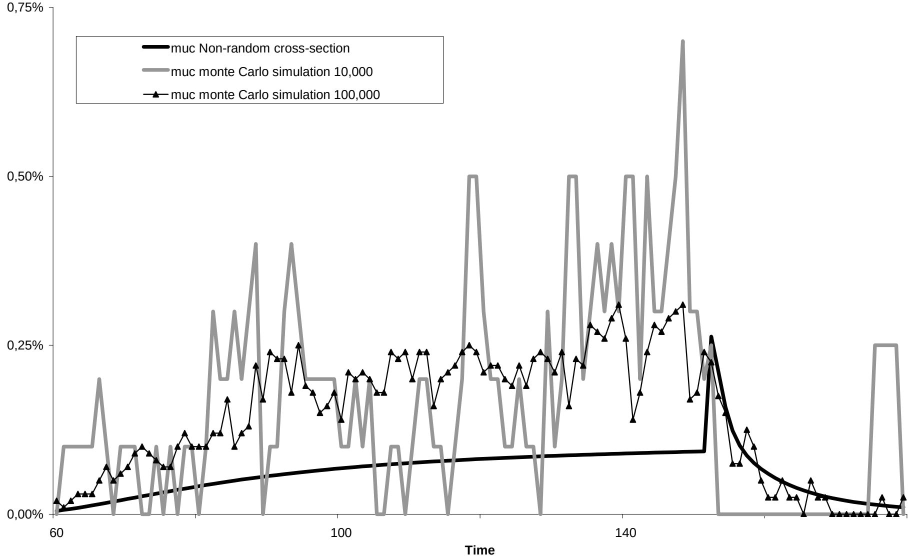
mu1 Monte Carlo simulation N=10,000
mu1 Monte Carlo simulation N=100,000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muc Non-random cross-section
muc monte Carlo simulation 10,000
muc monte Carlo simulation 100,000
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140
Time

Imposing equilibrium

- In model above, equilibrium is automatically imposed in simulation
- Why?

Imposing equilibrium

- What if we add one-period bonds?
  - Also solve for
    - individual demand for bonds, $b(s_{i,t})$
    - bond price, $q(s_t)$
  - Simulated aggregate demand for bonds not necessarily $= 0$
  - Why is this problematic?

Bonds and ensuring equlibrium I

- Add the bond price as a state variable in individual problem
    - a bit weird (making endogenous variable a state variable)
    - risky in terms of getting convergence

Bonds and ensuring equlibrium II

- Don't solve for

$$b_i(s_{i,t})$$

- but solve for

$$b_i(q_t, s_{i,t})$$

  - where dependence on q_t comes from an equation

- Solve q_t from

$$0 = \left(\sum_i^I b_i(q_t, s_{i,t}) \right) / I$$

Bonds and ensuring equlibrium II

How to get $b_i(q_t, s_{i,t})$?

1. Solve for $d_i(s_{i,t})$ where

$$d(s_{i,t}) = b(s_{i,t}) + q(s_t)$$

   - this adds an equation to the model

2. Imposing equilibrium gives

$$\begin{aligned} 0 &= \left(\sum b_i(q_t, s_{i,t})\right)/I \quad \Longrightarrow \\ q_t &= \left(\sum d_i(s_{i,t})\right)/I \\ b_{i,t+1} &= d(s_{i,t}) - q(s_t) \end{aligned}$$

Bonds and ensuring equlibrium II

- Does any $b_i(q_t, s_{i,t})$ work?
- For sure it needs to be a demand equation, that is

$$\frac{\partial b_i(q_t, s_{i,t})}{\partial q_t} < 0$$

Bonds and ensuring equlibrium II

Many ways to implement above idea:

- $d(s_{i,t}) = b(s_{i,t}) + q(s_t)$ is ad hoc (no economics)
- Alternative:
  - solve for $c(s_{i,t})$
  - get $b_{i,t}$ from budget constraint which contains q_t
  - You get $b_i(q_t, s_{i,t})$ with

$$\frac{\partial b_i(q_t, s_{i,t})}{\partial q_t} < 0$$

KS algorithm: Advantages & Disadvantages

- simple
- MC integration to calculate cross-sectional means
  - can easily be avoided
- Points used in projection step are clustered around the mean
  - Theory suggests this would be bad (recall that even equidistant nodes does not ensure uniform convergence; Chebyshev nodes do)
  - At least for the model in KS (1998) this is a non-issue; in comparison project the aggregate law of motion for K obtained this way is the most accurate

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