Solving Models with Heterogeneous Agents Applications

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Models considered

- Monetary models with consumer heterogeneity
- Models with entrepreneurs
  - discrete choice
- Turning KS into a matching model
- Portfolio problem

Monetary model I: simple MIU model

- Endowment economy
- Idiosyncratic and aggregate endowment shocks
- Agents can hold money to ensure themselves
- Money also provides liquidity services
- Equilibrium model but constant money supply (for now)

Exogenous processes

Endowment process:

$$y_{i,t} = a_t \theta \exp\left(e_{i,t}\right)$$

$$a_t = (1-\rho_a) + \rho_a a_{t-1} + \varepsilon_{a,t}, \quad \varepsilon_{a,t} \sim N(0, \sigma_a^2)$$
$$e_{i,t} = \rho_e e_{i,t-1} + \varepsilon_{e,i,t} \sim N(0, \sigma_e^2)$$

$$\int y_{i,t} di = a_t$$

Household problem

$$\max_{\{c_{i,t},M_{i,t}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \left[\ln c_{i,t} + \mu \ln \frac{M_{i,t}}{p_t} \right]$$

s.t.

$$P_t c_{i,t} + M_{i,t} = P_t y_{i,t} + M_{i,t-1}$$

$M_{i,0}$ given

First-order condition

$$\frac{1}{P_t}\frac{1}{c_{i,t}} = \mathsf{E}_t\left[\beta\frac{1}{P_{t+1}}\frac{1}{c_{i,t+1}} + \frac{\mu}{M_{i,t}}\right]$$

Equilibrium

$$\int M_{i,t} di = \int \overline{M} di = \overline{M}$$

(note that there is a unit mass of agents)

Getting started

- Give the true set of state variables
  - they have to be known at the beginning of the period
- Which cross-sectional moments would be included if you include second-order moments
- Why would $\rho_e = 0$ reduce the computational burden?

Quick (and dirty) solution

Procedure

1. Solve the rep. agent model
2. Use its law of motion to obtain solution to our model

Rep. agent

$$\frac{1}{P_t}\frac{1}{a_t} = \mathsf{E}_t\left[\beta\frac{1}{P_{t+1}}\frac{1}{a_{t+1}} + \frac{\mu}{M}\right]$$

Alternative strategy

- Start with $P_t = \gamma_0 + \gamma_1 a$
- Using this one can solve for $M_{i,t}$ from

$$\begin{aligned}
\frac{1}{P_t}\frac{1}{c_{i,t}} &= \mathsf{E}_t\left[\beta\frac{1}{P_{t+1}}\frac{1}{c_{i,t+1}} + \frac{\mu}{M_{i,t}}\right] \\
P_t c_{i,t} + M_{i,t} &= P_t y_{i,t} + M_{i,t-1} \\
P_t &= \gamma_0 + \gamma_1 a
\end{aligned}$$

Problem

- In simulation you will never get

$$\int M_{i,t} = \overline{M}$$

  even if approximation for P_t is quite good
- Even worse, errors are likely to accumulate $\Longrightarrow$ Disequilibrium is accumulating over time
- Not the type of errors you want to live with

Solution to the problem

- Solve for $M_i(P_t, e_{i,t}, a_t)$ instead of $M_i(e_{i,t}, a_t)$
  dependence on P_t must come from "model" equation
- Each period solve for P_t from

$$\int M_i(P_t, e_{i,t}, a_t) = \overline{M}$$

Solution to the problem

- Which $M_i(P_t, e_{i,t}, a_t)$ to choose?
- Lots probably work
  - !!! But you do need

$$\frac{\partial M_i(P_t, \cdot)}{\partial P_t} > 0$$

Procedure

- Start with $P_t = \gamma_0 + \gamma_1 a$
- In addition to the model

$$\begin{aligned}\frac{1}{P_t}\frac{1}{c_{i,t}} &= \mathsf{E}_t\left[\beta\frac{1}{P_{t+1}}\frac{1}{c_{i,t+1}} + \frac{\mu}{M_{i,t}}\right] \\ P_t c_{i,t} + M_{i,t} &= P_t y_{i,t} + M_{i,t-1} \\ P_t &= \gamma_0 + \gamma_1 a\end{aligned}$$

define $P_{i,t}$ using

$$M_{i,t} = P_t + \overline{M} - P_{i,t}$$

Procedure

- This gives following numerical solutions:
  - $c_i(e_{i,t},a_t), M_i(e_{i,t},a_t), P_i(e_{i,t},a_t)$
- Do **not** use:
  - $c_i(e_{i,t},a_t), M_i(e_{i,t},a_t)$
- Only use

$$M_{i,t} = P_t + \overline{M} - P(e_{i,t},a_t)$$

Procedure

- Solve for P_t from

$$\int M_{i,t} di = \int \left(P_t + \overline{M} - P(e_{i,t}, a_t) \right) di$$

- Since

$$\int M_{i,t} di = \int \overline{M} di$$

we get

$$P_t = \int P_{i,t} di$$

Procedure

- Update γ_0 and γ_1
  - With Xpa coefficients would follow directly from

$$P_t = \int P_{i,t} di$$

  - With KS you would run a regression

Model with entrepreneurs - discrete choice

- ex ante identical agents
- perpetual youth model (constant probability of dying)
- agents can buy ability to invest in more productive capital (become entrepreneurs)
- all agents work (for simplicity)

Representative firm

$$
\begin{aligned}
Y_t &= a_t \left(K_t + K_{e,t}\right)^{\alpha} L_t^{1-\alpha} \\
r_t &= \alpha a_t \left(K_t + K_{e,t}\right)^{\alpha-1} L_t^{1-\alpha} \\
w_t &= (1-\alpha) a_t \left(K_t + K_{e,t}\right)^{\alpha} L_t^{-\alpha}
\end{aligned}
$$

Entrepreneur

$$
\begin{aligned}
v_e(e_{i,t}k_{i,t}, a_t) &= \max_{k_{i,t+1}} \left(\begin{array}{c} \ln(r_t e_{i,t} k_{i,t} + (1-\delta) e_{i,t} k_{i,t} + w_t - k_{i,t+1}) \\ +\beta(1-\rho)\mathsf{E}_t\left[v_e(k_{i,t+1}, a_{t+1})\right] \end{array} \right) \\
e_{i,t} &\sim N(1+\mu, \sigma_e^2), \quad \mu > 0
\end{aligned}
$$

Entrepreneur - FOC

$$\frac{1}{c_{i,t}} = \mathsf{E}_t \left[\frac{\beta e_{i,t+1} \left(r_{t+1} + (1-\delta) \right)}{c_{i,t+1}} \right]$$

Non-entrepreneur

$$v(\varepsilon_{i,t}k_{i,t},a_t)$$
$$=$$
$$\max\left\{\begin{array}{c} \max\limits_{k_{i,t+1}}\left(\begin{array}{c}\ln(r_t\varepsilon_{i,t}k_{i,t}+(1-\delta)\varepsilon_{i,t}k_{i,t}+w_t-k_{i,t+1}) \\ +\beta(1-\rho)\mathsf{E}_t\left[v(k_{i,t+1},a_{t+1})\right]\end{array}\right) \\ \\ \max\limits_{k_{i,t+1}}\left(\begin{array}{c}\ln(r_t\varepsilon_{i,t}k_{i,t}+(1-\delta)\varepsilon_{i,t}k_{i,t}+w_t-k_{i,t+1}-\psi) \\ +\beta(1-\rho)\mathsf{E}_t\left[v_e(k_{i,t+1},a_{t+1})\right]\end{array}\right)\end{array}\right\}$$

$$\varepsilon_{i,t}\sim N(1,\sigma_e^2)$$

Non-entrepreneur - FOC

$$\frac{1}{c_{i,t}} = \mathsf{E}_t \left[\frac{\beta \varepsilon_{i,t+1} \left(r_{t+1} + (1 - \delta) \right)}{c_{i,t+1}} \right]$$

level $\overline{\varepsilon k}_t$ pinned down by

$$\max_{k_{i,t+1}} \left(\begin{array}{c} \ln(r_t\overline{\varepsilon k}_t + (1-\delta)\overline{\varepsilon k}_t + w_t - k_{i,t+1}) \\ +\beta(1-\rho)\mathsf{E}_t\left[v(k_{i,t+1}, a_{t+1})\right] \end{array} \right)$$

$$=$$

$$\max_{k_{i,t+1}} \left(\begin{array}{c} \ln(r_t\overline{\varepsilon k}_t + (1-\delta)\overline{\varepsilon k}_t + w_t - k_{i,t+1} - \psi) \\ +\beta(1-\rho)\mathsf{E}_t\left[v_e(k_{i,t+1}, a_{t+1})\right] \end{array} \right)$$

Equilibrium

$$
\begin{aligned}
N_{e,t+1} &= (1-\rho)N_{e,t} + F_t(\overline{\varepsilon k}_t) \\
K_t &= N_{e,t}\int k dF_{e,t} + (1-N_{e,t})\int k dF_{e,t} \\
L_t &= 1
\end{aligned}
$$

Getting started

- Give the true set of state variables
- How to deal $F_t(\overline{\varepsilon k_t})$?
- With which aggregate laws of motion could you solve the individual problem?

The KS model with matching

- Unemployment rate is exogenous in KS
- More interesting would be to make this endogenous using a matching model

Individual agent

- Subject to employment shocks ($e_{i,t} \in \{0, 1\}$)
- constant wage rate and no unemployment insurance
- Incomplete markets
    - only way to save is through holding ownership shares
    - penalty function on number of shares

Laws of motion

- a_t can take on two values
- e_t can take on two values as in original KS model
  - but probabilities determined endogenously

Individual agent

$$\max_{\{c_{i,t},s_{i,t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t\left(\ln(c_{i,t})-F(s_{i,t+1})\right)$$

s.t.

$$c_{i,t}+s_{i,t+1}p_t=we_{i,t}+s_{i,t}\left(p_t+d_t\right)$$

- simple problem for given process of r_t and matching probabilities

Penalty function

$$F(s_{i,t+1}) = \frac{\eta_1}{\eta_0} \exp(-\eta_0 s_{i,t+1}) + \eta_1 \exp(1) s_{i,t+1}$$

$$f(s_{i,t+1}) = \frac{\partial F(s_{i,t+1})}{\partial s_{i,t+1}} = -\eta_1 \exp(-\eta_0 s_{i,t+1}) + \eta_1 \exp(1)$$

Minimum of penalty function

$$f(s_{i,t+1}) = \frac{\partial F(s_{i,t+1})}{\partial s_{i,t+1}} = 0 \text{ at } s_{i,t+1} = 1$$

- changes in $s_{i,t+1}$ costly if η_0 high
- If η_0 high enough, then
  - $s_{i,t+1} \approx 1$
  - $c_{i,t} \approx we_{i,t} + d_t$

Individual agent - FOC condition

$$\frac{p_t}{c_{i,t}} + f(s_{i,t+1}) = \mathsf{E}_t\left[\frac{\beta\left(p_{t+1} + d_{t+1}\right)}{c_{i,t+1}}\right]$$

Law of motion of employment status

$$\text{prob}(e_{+1} = 1 | e = 1) = 1 - \rho$$

$$\text{prob}(e_{+1} = 1 | e = 0) = \pi_t$$

π_t is endogenous

Law of motion of employment status

$$\text{prob}(e_{+1} = 1 | e = 1) = 1 - \rho$$

$$\text{prob}(e_{+1} = 1 | e = 0) = \pi_t$$

π_t is endogenous

Representative firm

$$\max_{\{n_{t+1},k_t,v_t\}_{t=1}^{\infty}} \mathsf{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \frac{c_1}{c_t} \left((a_t - w)\, n_t - \psi v_t \right) \right]$$

s.t.

$$n_{t+1} = \tilde{\pi}_t v_t + (1-\rho_x) n_t$$

- $\tilde{\pi}_t$: matching probability vacancy
- MRS used is ad hoc. Why?

Representative firm - FOC

$$\psi = \tilde{\pi}_t \lambda_t$$

$$\lambda_t = \mathsf{E}_t \left[\beta \left(a_{t+1} - w + (1 - \rho_x) \lambda_{t+1} \right) \right]$$

Equilibrium and matching market

$$\int s_{i,t} di = 1 \text{ or } c_t = \int c_{i,t} di = w_t n_t + d_t$$

$$m_t = \nu_1 (1 - n_t)^{\nu_2} v_t^{1-v_2}$$

$$\tilde{\pi}_t = \frac{m_t}{v_t} \ \& \ \pi_t = \frac{m_t}{1 - n_t}$$

Getting started

- Give the true set of state variables
- which laws of motions for aggregate variables do you need
- how connected are the different sectors?

Static portfolio problem

$$\max_{\alpha} \mathsf{E}\left[U(\alpha r_1 + (1-\alpha) r_2\right]$$

$$r_1 \sim N(\mu, \sigma_1^2)$$
$$r_2 \sim N(\mu, \sigma_2^2)$$

Steady state value of α (i.e. value when $\sigma_1 = \sigma_2 = 0$) is not defined

Devereux-Sutherland trick

Solve α from second-order approximation

$$
\begin{gathered}
[U(\alpha r_1 + (1-\alpha) r_2] \\
\approx \\
U(\mu) \\
+\alpha U'(\mu)(r_1 - r_2) \\
+0.5\alpha^2 U''(\mu)((r_1-\mu)^2 + (r_2-\mu)^2)
\end{gathered}
$$

Devereux-Sutherland trick

Solve α from second-order approximation

$$\max_{\alpha} \mathrm{E}0.5(\alpha^2(r_1-\mu)^2+(1-\alpha)^2(r_2-\mu)^2)$$

$$\alpha = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

But is this a difficult problem?

- Using standard projection techniques this is a trivial problem

$$\max_{\alpha} \frac{\sum_{i_1}^{I} \sum_{i_2}^{I} U \begin{pmatrix} \alpha \left(\sigma_1 \sqrt{2} \zeta_{i_1} \right) \\ +(1-\alpha) \left(\sigma_2 \sqrt{2} \zeta_{i_2} \right) \end{pmatrix} \omega_1 \omega_2}{\sqrt{\pi}}$$

Making portfolio problem well behaved

- Portfolio problems remain a bit tricky
- Start with a problem with frictions so that problem is trivial to solve
- Gradually decrease friction
- Ask whether friction should be zero to answer your question

Making portfolio problem well behaved

$$\max_{\alpha} -\eta_0 (\alpha - \bar{\alpha})^2 + \mathsf{E}\left[U(\alpha r_1 + (1-\alpha) r_2\right]$$

- Problem easier when $\eta_0 > 0$
- Many insights same for $\eta_0 = 0$ and $\eta_0 > 0$
- For example

$$\frac{\partial \alpha}{\partial \sigma_1} < 0$$

Dynamic portfolio problem

- log endowment

$$y_t = \rho_y y_{t-1} + e_{y,t} \quad e_{y,t} \sim N(0, \sigma_y^2)$$

- returns

$$\begin{aligned} r_f &\sim N(\mu_{r_f}, 0) \\ r &\sim N(\mu_r, \sigma_r^2) \end{aligned}$$

- Budget constraint

$$c_t + s_t = y_t + \left(1 + \alpha_{t-1} r_f + (1 - \alpha_{t-1}) r_t\right) s_{t-1}$$

Maximization problem

$$\max_{\{\alpha_t, s_t, c_t\}} \sum_{t=1}^{\infty} \beta^{t-1} \left(\begin{array}{c} \ln(c_t) - \frac{\theta_0}{2}(\alpha_t - 0.5)^2 \\ -\frac{\eta_1}{\eta_0} * \exp(-\eta_0 s_t) - \eta_1 s_t \end{array} \right)$$

- first-order conditions

$$\frac{1}{c_t} = \mathsf{E}\left[\frac{\beta\left(1 + \alpha_t r_f + (1-\alpha_t) r_{t+1}\right)}{c_{t+1}}\right] + \eta_1\left(\exp\left(-\eta_0 s_t\right) - s_t\right)$$

$$0 = \mathsf{E}\left[\frac{r_f - r_{t+1}}{c_{t+1}}\right] - \theta_0\left(\alpha_t - 0.5\right)$$

- How to start?

References

- Devereux, M.B., and A. Sutherland, 2010, Country Portfolio Dynamics, Journal of Economic Dynamics.