

Solving Models with Heterogeneous Agents (not with KS, REITER or Xpa)

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Algorithms other than KS, REITER, & XPA

- Den Haan (1996)
 - very simple
- Roca & Preston (2007)
 - pure perturbation, thus fast
- Algan, Allais, & Den Haan (2008)
 - pure projection; can handle transition from non-typical cross-sectional distributions

Den Haan 1996

- Cross-sectional distribution characterized with finite set of moments
- No *explicit* approximate law of motion for aggregate variables
 - \implies no additional inaccuracies introduced
- Full simulation method
 - \implies easy
 - \implies computationally expensive

$$c_{i,t}^{1-\nu} = p_N(s_{i,t}; \psi)$$

solve $k_{i,t+1}$ from budget constraint

$$\begin{array}{l} \text{if } k_{i,t+1} \geq 0 \\ \text{if } k_{i,t+1} < 0 \end{array} \left\{ \begin{array}{l} \text{done} \\ k_{i,t+1} = 0 \\ \text{solve } c_{i,t} \text{ from bc} \end{array} \right.$$

- Conditional expectation of *individual* $\approx p_N(s_{i,t}; \psi)$
- $s_{i,t} = \{k_{i,t}, e_{i,t}, z_t, \text{info about cross-sectional distribution}\}$

Simulating a panel for given value ψ

- generate aggregate productivity $\{z_t\}_{t=1}^T$
- start in $t = 1$ with cross-section of I agents
 - Thus, $k_{i,t}$ and $e_{i,t}$ known at $t = 1$
- use cross-section to calculate K_t and other moments
- use K_t and z_t to calculate r_t and w_t
- for each agent calculate $k_{i,t+1}$
- for each agent draw new $e_{i,t}$ and go to the next period

Simulating a panel for given value ψ

- To update individual problem:
 - you only need to variables for 1 agent
- But individual choices depend on aggregates
 - \implies you need a panel

Updating individual law of motion

$$\text{If } k_{t+1} > 0 \quad \left\{ \begin{array}{l} c_t^{-\nu} = E_t [\beta c_{t+1}^{-\nu} (r_{t+1} + 1 - \delta)] \\ \text{solve } k_{t+1} \text{ from budget constraint} \end{array} \right.$$

- collect observations with $k_{t+1} > 0$
- regress $\beta c_{t+1}^{-\nu} (r_{t+1} + 1 - \delta)$ on $p_n(s_t; \psi) \implies \hat{\psi}$
- aggregate law of motion taken care of (implicitly)
- update ψ using weighted average of $\hat{\psi}$ and old ψ

Updating aggregate law of motion

- Not needed
- In the simulation, aggregate variables are constructed by explicitly aggregating the values across I individuals
- Thus, no approximation needed to describe law of motion of aggregate variables

Advantages & Disadvantages

- even simpler than KS
- even construction of individual policy rules done using simulation methods
 - disadvantages of simulation methods can be lessened/avoided using the improvements suggested by Maliar, Maliar, & Judd

Imposing equilibrium in panel

- It is important to impose equilibrium
 - unless the average deviation (across periods) is exactly zero the deviation will accumulate and increase without bound
- True for any simulation procedure (including KS)
- Equilibrium automatically imposed in capital economy

Tricks to impose equilibrium

Suppose, you want to impose

$$\sum_{i=1}^I b(s_{i,t}) = 0$$

Then, get approximation for $d(s_{i,t})$, where

$$d(s_{i,t}) = q_t - b(s_{i,t})$$

and get q_t from

$$q_t = \sum_{i=1}^I d(s_{i,t}) / I_t$$

and $b_{i,t}$ from

$$b_{i,t} = q_t - d(s_{i,t})$$

Tricks to impose equilibrium

You may think that

$$d(s_{i,t}) = q_t - b(s_{i,t})$$

and

$$d(s_{i,t}) = q_t + b(s_{i,t})$$

both work.

But stability properties of the algorithm can be very different

Tricks to impose equilibrium

$$\text{If } \frac{\sum_{i=1}^I b_i}{I} = \bar{b} > 0,$$

then the bond price is too low.

$$\text{If } d(s_{i,t}) = q_t - b(s_{i,t})$$

then

$$q^{\text{new}} = \frac{\sum_{i=1}^I d(s_{i,t})}{I} = \frac{\sum_{i=1}^I q^{\text{old}}}{I} - \bar{b}$$

and

$$q^{\text{new}} > q^{\text{old}}$$

as needed

Pure perturbation (Roca & Preston)

- KS model has to be modified a little
 - discrete support is likely to be difficult \implies continuous support
 - borrowing constraint is definitely difficult \implies penalty function
- Perturbation around solution when there is no aggregate and no idiosyncratic uncertainty
- Capital is in levels (not in logs or any other transformation)

Model modifications

- $e_{i,t}$ can take on continuum of values

$$e_{i,t+1} = (1 - \rho_e) + \rho_e e_{i,t} + \varepsilon_{i,t+1}^e$$

$$\varepsilon_{i,t+1}^e \sim N(0, \sigma^2)$$

$$\mathbb{E}e_{i,t} = 1 \implies L = 1$$

- Continuous penalty term when capital is getting smaller. FOCs:

$$c_{i,t}^{-\nu} = \mathbb{E}_t \left[-2\phi k_{i,t+1}^{-3} + c_{i,t+1}^{-\nu} (r_{t+1}^k + 1 - \delta) \right]$$

$$k_{i,t+1} = (1 - \delta)k_{i,t} + r_t k_{i,t} + w_t e_{i,t} \bar{l} - c_{i,t}$$

State variables and order of perturbation

- Individual state variables $s_{i,t} = \{k_{i,t}, e_{i,t}, \tilde{s}_t\}$
- Again a limited set of moments is used as state variables
- As with Xpa, the elements of \tilde{s}_t depend on approximation order
 - First order: $\tilde{s}_t = \{a_t, K_t\}$
 - Second order: $\tilde{s}_t = \{a_t, K_t, \Phi_t, \Psi_t\}$
 -

$$K_t = \int_0^1 (k_{i,t} - K_t) di,$$

$$\Phi_t = \int_0^1 (k_{i,t} - K_t)^2 di, \text{ and}$$

$$\Psi_t = \int_0^1 (k_{i,t} - K_t) (e_{i,t} - \mu_e) di.$$

What to solve for?

Solve for $h_v(s_{i,t}, \sigma)$ with $v \in \{c, k, K, \Phi, \Psi\}$

What to solve for?

$$\frac{1}{h_c(s_{i,t}, \sigma)} = \beta E_t \left[-2\phi h_k(s_{i,t}, \sigma)^{-3} + \frac{(r_{t+1} + 1 - \delta)}{h_c(s_{i,t+1})} \right]$$

$$h_k(s_{i,t}, \sigma) = (1 - \delta)k_{i,t} + r_t k_{i,t} + w_t e_{i,t} \bar{l} - h_c(s_{i,t}, \sigma)$$

$$K_{t+1} = h_K(\tilde{s}_t, \sigma) = \int_0^1 h_k(s_{i,t}, \sigma) di$$

$$\Phi_{t+1} = h_\Phi(\tilde{s}_t, \sigma) = \int_0^1 (h_k(s_{i,t}, \sigma) - \bar{k})^2 di$$

$$\Psi_{t+1} = h_\Psi(\tilde{s}_t, \sigma) = \int_0^1 (h_k(s_{i,t}, \sigma) - \bar{k}) (e_{i,t} - \mu_e) di$$

$$r_t = \alpha a_t (K_t/\bar{l})^{\alpha-1}$$

$$w_t = (1 - \alpha)a_t (K_t/\bar{l})^{\alpha}$$

$$s_{i,t+1} = \left\{ \begin{array}{l} h_k(s_{i,t}), 1 - \rho_e + \rho_e e_{i,t} + \varepsilon_{i,t+1}, 1 - \rho + \rho a_t + \varepsilon_{a,t+1}, \\ h_K(\tilde{s}_t, \sigma), h_{\Phi}(\tilde{s}_t, \sigma), h_{\Psi}(\tilde{s}_t, \sigma) \end{array} \right\}$$

- system expressed in period t variables and period $t + 1$ shocks
- we now have a perturbation system
 - differentiating system gives values derivatives of h_v
 - from these we get coefficients of Taylor expansions

Comments

- There is no approximation in system specified so far (which is good)
- w_t and r_t depend on mean of the *level* of capital
- We have an exact equation for the mean *because capital is in levels* (and not in logs)
- Capital in logs \implies one has to approximate aggregation definition
 - e.g., linearizing around steady state
 - given dispersion in capital levels this may not be accurate

Moments and order of perturbation (first-order)

- Agents obviously care about $k_{i,t}$, $e_{i,t}$, a_t , and K_{t+1}
- First-order perturbation \implies linear policy rules $\implies K_{t+1}$ only depends on K_t , a_t , and nothing else (mean of $e_{i,t}$ is constant through time)

Moments and order of perturbation (second-order)

- Agents obviously care about $k_{i,t}$, $e_{i,t}$, a_t , and K_{t+1}
- Second-order perturbation \implies agent's policy rules depend on $(k_{i,t} - \bar{k})^2$ and $(k_{i,t} - \bar{k})(e_{i,t} - \mu_e)$
 - $\implies K_{t+1}$ depends on K_t , Φ_t , Ψ_t , and z_t , and nothing else
- Does second-order perturbation include terms like
 - $(k_{i,t} - \bar{k})^2 K_t$?
 - $K_t \Psi_t$?
 - $\Psi_t \Phi_t$?

Dealing with transitions

- KS, Xpa, Den Haan 1996, Roca Preston, & Reiter (hybrid) focus on
 - small changes aggregate variables close to steady state
 - behavior aggregate variables in a *typical* simulation
- This means they cannot deal with
 - transition after a *one-time* and *unforeseen* redistribution of capital
 - destruction of capital

Simple analogy

- Projection solution of the neoclassical growth model gives

$$k' \approx p_N(k, a; \psi)$$

- By using a wide enough grid for k and a and a rich enough approximating function one ensures accuracy for *all* values of k and a inside the grid including those not encountered in a simulation
- In solving heterogeneous agent models can you attain accuracy for any cross-sectional distribution?

Algan, Allais, and Den Haan (2008)

- use projection methods and quadrature techniques as much as possible
 - \implies construct a grid for the aggregate state variables including moments
- calculate next-period's moments using quadrature techniques
 - quadrature integration requires functional form for the distribution
 - use flexible functional form and link moments with polynomial's coefficients

AAD - Key step

- moments \iff functional form density
- trivial for Normal
- can this be generalized?

AAD - Flexible cross-sectional density

$$P_{N_m}(k, \rho) = \rho_0 \exp \left(\begin{array}{c} \rho_1 [k - m(1)] + \\ \rho_2 [(k - m(1))^2 - m(2)] + \dots + \\ \rho_{N_m} [(k - m(1))^{N_m} - m(N_m)] \end{array} \right).$$

where $m(n)$ is the n^{th} uncentered moment

- Goal: find the ρ s such that values of moments match implied moments
- This particular functional form \implies

$$\min_{\rho_1, \rho_2, \dots, \rho_{N_m}} \int_0^{\infty} P_{N_m}(k, \rho) dk.$$

and this is a convex problem

AAD #1: specify aggregate law of motion

- construct grid for aggregate state variables
- calculate next period's moments
- do projection step to calculate aggregate law of motion
- construct grid for individual agent (includes aggregate state variables)
- solve individual problem (for given aggregate law of motion)
- iterate between aggregate and individual problem

AAD #2: do not specify aggregate law of motion

- construct grid for individual problem (includes aggregate state variables)
- calculate next period's mean (and thus r_t and w_t) directly using quadrature techniques
- Solve individual problem

Problem with algorithm so far

- Moments fulfill two roles
 - state variable
 - get the shape of the distribution right
- To get shape right, you need several moments
 - \implies you need several state variables
- Solution:
 - use limited set of moments as state variables
 - use additional higher-order moments as reference moments for good shape

Getting the distribution right

- Suppose you only use the mean capital stock as a state variable
- But use $N(K, \sigma_K^2)$ as the cross-sectional distribution
- You still have to find σ_K^2
- AAD get reference moments from a simulation
- reference moments could depend on the state, i.e.,
 $N(K, (\sigma_K(\tilde{s}_t))^2)$

References

- Algan, Y., O. Allais, W.J. Den Haan, P. Rendahl, 2010, Solving and simulating models with heterogeneous agents and aggregate uncertainty
 - survey article available online with references to approaches discussed here
- Judd, K. L. Maliar, and S. Maliar, 2011, One-node quadrature beats Monte Carlo: A generalized stochastic simulation algorithm, NBER WP 16708
 - stresses importance of using $E_t [y_{t+1}]$ instead of y_{t+1} in PEA
- Judd, K. L. Maliar, and S. Maliar, 2010, Numerically stable stochastic methods for solving dynamics models, NBER WP 15296
 - give several useful suggestions to make PEA better