Learning Sunspots in Nonlinear Models

Wouter J. Den Haan
London School of Economics

July 24, 2014

Overview

- Key question
- Particular model
- Analysis for linearized model
- Algorithm for true nonlinear model

Rational expectations equilibrium

- Let $g(x_t, \zeta_t; \eta^*, \sigma_\zeta^*)$ be a rational expectations solution, where
    - x_t is a vector with the usual state variables
    - ζ_t is the sunspot variable
        - with $\mathrm{E}_t\left[\zeta_{t+1}\right] = 0$ and $\mathrm{E}_t\left[\zeta_{t+1}^2\right] = \left(\sigma_\zeta^*\right)^2$
    - η^* are the function's coefficients

Beliefs

- Agents' expectations are based on the belief that

$$g(x_t, \zeta_t; \eta^*, \sigma_\zeta^*) = g(x_t, \zeta_t; \eta_{\text{perceived}}, \sigma_{\zeta,\text{perceived}})$$

- $\eta_{\text{perceived}}$ and $\sigma_{\zeta,\text{perceived}}$ are the coefficients of $g(\cdot)$
- agents are assumed to use the correct functional form !!!
    - framework modified below to let agents *approximate* $g(\cdot)$

Behavior with non-REE beliefs

- **Model** is such that if expectations are based on

$$g(x_t,\zeta_t;\eta_{\text{perceived}},\sigma_{\zeta,\text{perceived}}),$$

  then actual behavior is given by

$$g(x_t,\zeta_t;\eta_{\text{actual}},\sigma_{\zeta,\text{actual}})$$

- **T-mapping**: This can be represented as

$$\begin{bmatrix} \eta_{\text{actual}} \\ \sigma_{\zeta,\text{actual}} \end{bmatrix} = T\left(\begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta,\text{perceived}} \end{bmatrix}\right)$$

Updating beliefs

- **Adaptive expectations**: Beliefs are updated iteratively using

$$\begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta,\text{perceived}} \end{bmatrix} = \begin{bmatrix} \eta_{\text{actual}} \\ \sigma_{\zeta,\text{actual}} \end{bmatrix}$$

or possibly

$$\begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta,\text{perceived}} \end{bmatrix} = (1-\omega) \begin{bmatrix} \eta_{\text{actual}} \\ \sigma_{\zeta,\text{actual}} \end{bmatrix} + \omega \begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta,\text{perceived}} \end{bmatrix}$$

Complete iterative system

iteration i is indicated with a superscript

$$\begin{bmatrix} \eta^{i}_{\text{actual}} \\ \sigma^{i}_{\zeta,\text{actual}} \end{bmatrix} = T\left(\begin{bmatrix} \eta^{i}_{\text{perceived}} \\ \sigma^{i}_{\zeta,\text{perceived}} \end{bmatrix}\right)$$

$$\begin{bmatrix} \eta^{i+1}_{\text{perceived}} \\ \sigma^{i+1}_{\zeta,\text{perceived}} \end{bmatrix} = \begin{bmatrix} \eta^{i}_{\text{actual}} \\ \sigma^{i}_{\zeta,\text{actual}} \end{bmatrix}$$

Possible key question

- Let η^* and σ_ζ^* be coefficients of rational expectations solution
- possible key question:

$$\lim_{i \longrightarrow \infty} \begin{bmatrix} \eta^i_{\text{perceived}} \\ \sigma^i_{\zeta,\text{perceived}} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \eta^* \\ \sigma^* \end{bmatrix}$$

for

$$\begin{bmatrix} \eta^1_{\text{perceived}} \\ \sigma^1_{\zeta,\text{perceived}} \end{bmatrix} \in I_{\eta^*,\sigma^*},$$

where I_{η^*,σ^*} is a neigborhood around $\left(\eta^*, \sigma_\zeta^*\right)$.

Agents cannot learn sunspot itself

- The sunspot variable, ζ_t, is chosen
  - Reason: agents cannot learn from system which variable from outside system can be added to system

Also cannot learn importance of sunspot

- Importance of sunspot, σ_ζ, is still undetermined
- Agents cannot learn that either
  - If $\sigma^1_{\zeta,\text{perceived}} = 0$, then agents will never converge to a $\sigma^*_\zeta > 0$
  - If $\sigma^1_{\zeta,\text{perceived}}$ is small, then agents will never converge to a large σ^*_ζ

Key question

- Key question:

$$\lim_{i \longrightarrow \infty} \begin{bmatrix} \eta^{i+1}_{\text{perceived}} \\ \\ \sigma^{i+1}_{\zeta,\text{perceived}} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \eta^* \\ \\ \sigma^* \end{bmatrix}$$

for

$$\left[\eta^{1}_{\text{perceived}}\right] \in I_{\eta^*}, \text{ and } \sigma^{1}_{\zeta,\text{perceived}} = \sigma^*_{\eta}$$

Important to distinguish

1. Stability of $g(\cdot)$
2. Stability of $T(\cdot)$

These are two different things

Stability of g(.)

- stability of $g(\cdot)$ is about stability of time series

$$\mathsf{E}\left[\lim_{t\to\infty} K_t\right] \overset{?}{\neq} \infty$$

- $g(\cdot)$ is stable since it is an REE
- $g(\cdot)$ is "more stable" for sunspots
  - sunspots are made possible by extra eigen values with modulus less than 1

Stability of T(.)

- stability of $T(\cdot)$ is about stability of policy function itself
- $T(\cdot)$ tends to be complex and not so intuitive
  - $T(\cdot)$ is typically nonlinear even if $g(\cdot)$ is linear
- $T(\cdot)$ is "less stable" for sunspots
  - this is the stability puzzle

Particular model

- McGough, Meng, & Xue (2011) or MMX:
  - simple RBC model with externality
  - for some parameter values the model has learnable sunspots
  - key is to use a *negative* capital externality

Firm's production function

$$Y_t = A_t K_t^a H_t^b$$
$$A_t = \Lambda_A \overline{K}_t^{\alpha - a} \overline{H}_t^{\beta - b}$$

where:

- K_t and H_t are firm level variables
- $\overline{K}_t$ and $\overline{H}_t$ are aggregate variables (taken as given by firm)
- negative capital externality: $\alpha < a$

Firm's first-order conditions

$$\begin{aligned} R_t &= aA_tK_t^{a-1}H_t^b \\ W_t &= bA_tK_t^aH_t^{b-1} \end{aligned}$$

In equilibrium: $K_t = \overline{K}_t$ and $H_t = \overline{H}_t$. Thus

$$\begin{aligned} R_t &= aA_tK_t^{a-1}H_t^b = a\Lambda_A K_t^{\alpha-1}H_t^{\beta} \\ W_t &= bA_tK_t^aH_t^{b-1} = b\Lambda_A K_t^{\alpha}H_t^{\beta-1} \end{aligned}$$

Household's first-order conditions

$$C_t^{-\nu} = \mathsf{E}_t \left[\rho \left(1 - \delta + R_{t+1} \right) C_{t+1}^{-\nu} \right]$$

$$\Lambda_H H_t^{\chi} = W_t C_t^{-\nu}$$

Complete model

$$
\begin{aligned}
C_t^{-\nu} &= \mathsf{E}_t\left[\rho\left(1-\delta+a\Lambda_A K_{t+1}^{\alpha-1}H_{t+1}\right)C_{t+1}^{-\nu}\right] \\
\Lambda_H H_t^{\chi} &= b\Lambda_A K_t^{\alpha} H_t^{\beta-1} C_t^{-\nu} \\
Y_t &= \Lambda_A K_t^{\alpha} H_t^{\beta} \\
Y_t &= C_t + K_{t+1} - (1-\delta)K_t
\end{aligned}
$$

Analytical solution steady state

1. $H_{ss} = K_{ss} = 1$

2. Choose Λ_A & Λ_H so that this is true
   - Λ_A & Λ_H do not affect the dynamics (only scale)

3. Solve for C_{ss} from budget constraint

Log-linearized system: Indeterminacy & sunspots

- with H_t substituted out, linearized system can be represented as follows:

$$\begin{array}{ll} \text{budget constraint:} & \widetilde{k}_{t+1} = d_k \widetilde{k}_t + d_c \widetilde{c}_t \\ \text{Euler equation:} & \widetilde{c}_t = b_k \widetilde{k}_{t+1} + b_c \mathsf{E}_t \left[\widetilde{c}_{t+1} \right] \end{array}$$

- Let $\lambda_{J,1}$ and $\lambda_{J,2}$ be the two Eigenvalues of J

Linearized system: Indeterminacy & sunspots

linearized solution can be represented as follows:

$$\begin{pmatrix} \widetilde{k}_{t+1} \\ \widetilde{c}_{t+1} \end{pmatrix} = J \begin{pmatrix} \widetilde{k}_t \\ \widetilde{c}_t \end{pmatrix} + \begin{pmatrix} 0 \\ F\zeta_{t+1} \end{pmatrix}$$

$$= \begin{pmatrix} d_k & d_c \\ -\frac{b_k d_k}{b_c} & \frac{1-b_k d_c}{b_c} \end{pmatrix} \begin{pmatrix} \widetilde{k}_t \\ \widetilde{c}_t \end{pmatrix} + \begin{pmatrix} 0 \\ F\zeta_{t+1} \end{pmatrix}$$

where $\mathsf{E}_t\left[\zeta_{t+1}\right] = 0$ and $\mathsf{E}_t\left[\zeta_{t+1}^2\right] = 1$

Linearized system: Indeterminacy & sunspots

- Let $\lambda_{J,1}$ and $\lambda_{J,2}$ be the two eigen values of J
- Using Jordan decomposition of J

$$\begin{pmatrix} \widetilde{k}_{t+1} \\ \widetilde{c}_{t+1} \end{pmatrix} = P \begin{bmatrix} \lambda_{J,1} & 0 \\ 0 & \lambda_{J,2} \end{bmatrix} P^{-1} \begin{pmatrix} \widetilde{k}_{t} \\ \widetilde{c}_{t} \end{pmatrix} + \begin{pmatrix} 0 \\ F\zeta_{t+1} \end{pmatrix}$$

Understanding indeterminacy

- suppose $\zeta_t = 0 \forall t$ (ignore sunspot for simplicity)
- given $\widetilde{k}_1$, value of $\widetilde{c}_1$ can still be arbitrarily chosen
- $\widetilde{k}_2$ follows from budget constraint
- $\widetilde{c}_2$ follows from Euler equation
- that is, we are simply solving forward
- If $|\lambda_{J,1}| < 1$ and $|\lambda_{J,2}| < 1$, then this will converge
    - easy to find parameters to satisfy this condition

Understanding indeterminacy

- If $|\lambda_{J,1}| > 1$ and $|\lambda_{J,2}| < 1$, then it must be true that

$$P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}'\begin{pmatrix} \widetilde{k}_1 \\ \widetilde{c}_1 \end{pmatrix} = 0$$

  to ensure that series don't explode

- this pins down $\widetilde{c}_1$ as a function of $\widetilde{k}_1$

Forming expectations

- There are many ways in which you can formulate expectations
- We follow MMX:
  - agents use $\widetilde{k}_{t-1}$, $\widetilde{c}_{t-1}$, & ζ_t to make forecasts
  - $\widetilde{c}_t$ is solved from Euler equation using
    - $\widehat{\mathrm{E}}_t\left[\widetilde{c}_t\right]$ instead of $\widetilde{c}_t$ to determine RHS

MMX system

budget constraint $\widetilde{k}_{t+1} = d_k \widetilde{k}_t + d_c \widehat{\mathsf{E}}_t \left[\widetilde{c}_t \right]$

Euler equation $\widetilde{c}_t = b_k \widetilde{k}_{t+1} + b_c \widehat{\mathsf{E}}_t \left[\widetilde{c}_{t+1} \right]$

Perceived law of motion and expectations

Perceived law of motion:

$$\widetilde{c}_t = A + B\widetilde{k}_{t-1} + D\widetilde{c}_{t-1} + F\zeta_t$$

Expectations:

$$\begin{aligned} \widehat{\mathsf{E}}_t\left[\widetilde{c}_t\right] &= A + B\widetilde{k}_{t-1} + D\widetilde{c}_{t-1} + F\zeta_t \\ \widehat{\mathsf{E}}_t\left[\widetilde{c}_{t+1}\right] &= A + B\widetilde{k}_t + D\mathsf{E}_t\left[\widetilde{c}_t\right] + F\widehat{\mathsf{E}}_t\left[\zeta_{t+1}\right] \\ &= A + B\widetilde{k}_t + D\mathsf{E}_t\left[\widetilde{c}_t\right] \end{aligned}$$

Perceived & actual law of motion

if

$$\begin{aligned}
\widetilde{k}_t &= d_k \widetilde{k}_{t-1} + d_c \widetilde{c}_{t-1} \\
\widetilde{k}_{t+1} &= d_k \widetilde{k}_t + d_c \widehat{\mathsf{E}} \left[\widetilde{c}_t \right] \\
\widehat{\mathsf{E}}_t \left[\widetilde{c}_t \right] &= A + B \widetilde{k}_{t-1} + D \widetilde{c}_{t-1} + F \zeta_t \\
\widehat{\mathsf{E}}_t \left[\widetilde{c}_{t+1} \right] &= A + B \widetilde{k}_t + D \widehat{\mathsf{E}}_t \left[\widetilde{c}_t \right] \\
\widetilde{c}_t &= b_k \widetilde{k}_{t+1} + b_c \widehat{\mathsf{E}}_t \left[\widetilde{c}_{t+1} \right]
\end{aligned}$$

then

Actual law of motion

$$\widetilde{c}_t = \begin{bmatrix} \left(b_c\left(1+D\right)+b_k d_c\right)A \\ b_k\left(d_k^2+d_c B+b_c B\left(d_k+D\right)\right) \\ b_k d_c\left(d_k+D\right)+b_c\left(Bd_c+D^2\right) \\ \left(b_k d_c+b_c D\right)F \end{bmatrix}' \times \begin{bmatrix} 1 \\ \widetilde{k}_{t-1} \\ \widetilde{c}_{t\_1} \\ \widetilde{\zeta}_t \end{bmatrix}$$

T-Mapping

$$T\begin{pmatrix} A \\ B \\ D \\ F \end{pmatrix} = \begin{pmatrix} \left(b_c\left(1+D\right)+b_k d_c\right)A \\ b_k\left(d_k^2+d_c B\right)+b_c B\left(d_k+D\right) \\ b_k d_c\left(d_k+D\right)+b_c\left(B d_c+D^2\right) \\ \left(b_k d_c+b_c D\right)F \end{pmatrix}$$

Rational Expectations Equilibrium

$$
\begin{aligned}
A &= 0 \\
B &= J_{21} = -\frac{b_k d_k}{b_c} \\
D &= J_{22} = \frac{1 - b_k d_c}{b_c} \\
F &= \text{anything}
\end{aligned}
$$

Check

$$T\,(\text{REE}) \stackrel{?}{=} \text{REE}$$

$$T\begin{pmatrix} 0 \\ -\frac{b_k d_k}{b_c} \\ \frac{1-b_k d_c}{b_c} \\ F \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 \\ -\frac{b_k d_k}{b_c} \\ \frac{1-b_k d_c}{b_c} \\ F \end{pmatrix}$$

- This is true
    - this doesn't say much except; just a check on calculations

T-mapping and sunspot

- $$\left.\frac{\partial T}{\partial F}\right|_{\text{REE}} = 1$$

  $\Longrightarrow$ the exact unit-root behavior implies that initial beliefs are simply confirmed
- *Exact* unit-root behavior is valid when
  - you are at the fixed point
  - no stochastics (use population moments)
  - exact linear model
- $\Longrightarrow$ in practice you should simply fix F and not iterate on it
  - $\Longrightarrow$ learning is about the non-sunspot coefficients

PEA and learning the sunspot

Overview of remaing material

1. Setting up PEA to generate first-order approximation
2. Use PEA to generate higher-order approximation

Model

$$
\begin{aligned}
C_t^{-\nu} &= \mathsf{E}_t \left[\rho \left(1 - \delta + a \Lambda_A K_{t+1}^{\alpha-1} H_{t+1}^{\beta} \right) C_{t+1}^{-\nu} \right] \\
\Lambda_H H_t^{\chi} &= b \Lambda_A K_t^{\alpha} H_t^{\beta-1} C_t^{-\nu} \\
K_{t+1} &= \Lambda_A K_t^{\alpha} H_t^{\beta} + (1-\delta) K_t - C_t
\end{aligned}
$$

PEA - first-order

$$\begin{aligned} C_t^{-\nu} &= \exp\left\{\eta_0 + \eta_k \ln\left(K_{t-1}/K_{ss}\right) + \eta_c \ln\left(C_{t-1}/C_{ss}\right) + \eta_\zeta \zeta_t\right\} \\ \Lambda_H H_t^{\chi} &= b\Lambda_A K_t^{\alpha} H_t^{\beta-1} C_t^{-\nu} \\ K_{t+1} &= \Lambda_A K_t^{\alpha} H_t^{\beta} + (1-\delta) K_t - C_t \end{aligned}$$

where

$$\exp\left\{\eta_0 + \eta_k \ln\left(K_{t-1}/K_{ss}\right) + \eta_c \ln\left(C_{t-1}/C_{ss}\right) + \eta_\zeta \zeta_t\right\} \\ \approx \mathsf{E}_t\left[\rho\left(1-\delta + a\Lambda_A K_t^{\alpha-1} H_t^{\beta}\right) C_{t+1}^{-\nu}\right]$$

How to find eta coefficients

- η_ζ has to be fixed as explained above
- η_0, η_k, and η_c can be used using regular PEA algorithm
- Note that

$$\ln C_t = -\frac{\eta_0}{\nu} - \frac{\eta_k}{\nu}\ln\left(\frac{K_{t-1}}{K_{ss}}\right) - \frac{\eta_c}{\nu}\ln\left(\frac{C_{t-1}}{C_{ss}}\right) - \frac{\eta}{\nu}{}_\zeta \zeta_t$$

$\Longrightarrow$ solution from linearized system can be used as initial conditions

How to find eta coefficients continued

Iterative scheme:

- η_0^i, η_k^i, and η_c^i : coefficients at i^{th} iteration
- Simulate K_t, H_t, C_t, and $Z_{t+1} = \rho\left(1 - \delta + a\Lambda_A K_{t+1}^{\alpha-1} H_{t+1}^{\beta}\right) C_{t+1}^{-\nu}$
- $$\widehat{\eta} = \underset{\eta_0,\eta_k,\eta_c}{\arg\min} \sum_{T_1}^{T} \left((z_{t+1}) - \exp\left\{ \begin{array}{c} \eta_0 + \eta_k \ln\left(K_{t-1}/K_{ss}\right) \\ +\eta_c \ln\left(C_{t-1}/C_{ss}\right) + \eta_\zeta \zeta_t \end{array} \right\} \right)^2$$
- $$\eta^{i+1} = (1-\omega)\,\widehat{\eta} + \omega\eta^i$$

How to find eta coefficients continued

Comments:

- You are not allowed to take logs to get a linear regression equation!
- $0 \leq \omega < 1$: dampening factor
    - may be needed to get convergence

PEA - general setup

- Let $S_t = \{K_{t-1}, C_{t-1}\}$
- Approximation used:

$$\mathsf{E}_t\left[\rho\left(1-\delta+a\Lambda_A K_t^{\alpha-1}H_t^{\beta}\right)C_{t+1}^{-\nu}\right]$$

$$\approx$$

$$h_S(S_t;\eta_S)+\widetilde{\eta}\sin\left(h_\zeta\left(\zeta_t,S_t;\eta_\zeta\right)\right)$$

PEA - approximating function

- $h_S(S_t;\eta_S)$: a flexible functional form
- η_S : coefficients of $h_S(\cdot)$
- $h_\zeta(\zeta_t, S_t;\eta_\zeta)$: a flexible functional form
- η_ζ : coefficients of $h_\zeta(\cdot)$
- $\widetilde{\eta}$: *FIXED* coefficient that determines maximum impact sunspot (since $|\sin| \leq 1$)
    - fixing $\widetilde{\eta}$ corresponds to fixing F and σ_ζ above

PEA - finding eta coefficients

- Exactly as before
- Just a more complex nonlinear regression problem

Advantage of additive approximation

$$\mathsf{E}_t\left[\rho\left(1-\delta+a\Lambda_A K_t^{\alpha-1}H_t^{\beta}\right)C_{t+1}^{-\nu}\right]=$$

$$\mathsf{E}_t\left[\begin{array}{c}\rho\left(1-\delta+a\Lambda_A K_t^{\alpha-1}H_t^{\beta}\right)\times h_S(S_{t+1};\eta_S)+\\ \rho\left(1-\delta+a\Lambda_A K_t^{\alpha-1}H_t^{\beta}\right)\times\widetilde{\eta}\sin\left(h_\zeta\left(\zeta_{t+1},S_{t+1};\eta_\zeta\right)\right)\end{array}\right]$$

$\Longrightarrow$ sunspot part is likely to have little effect on $\mathsf{E}_t\left[\cdot\right]$

-
    - this mimics linear case
    - but in non-linear case $\mathsf{E}_t\left[\widetilde{\eta}\sin\left(h_\zeta\left(\zeta_{t+1},S_{t+1};\eta_\zeta\right)\right)\right]$ does not have to be zero, i.e., sun'spot can have first-order effects

A bit more on eta-tilde

- Approximation used is still:

$$\mathsf{E}_t\left[\rho\left(1-\delta+a\Lambda_A K_t^{\alpha-1}H_t^{\beta}\right)C_{t+1}^{-\nu}\right]$$
$$\approx h_S(S_t;\eta_S)+\widetilde{\eta}\sin\left(h_\zeta\left(\zeta_t,S_t;\eta_\zeta\right)\right)$$

- But go back to 1st-order approximation:

$$h_\zeta\left(\zeta_t,S_t;\eta_\zeta\right)=\eta_\zeta\zeta_t$$

A bit more on eta-tilde

- Question: What is

$$\lim_{i \longrightarrow \infty} \eta_\zeta^i \quad ?$$

- Exact unit-root type of linear non-stochastic setting not true
  - $\Longrightarrow$ unlikely that $\lim_{i \longrightarrow \infty} \eta_\zeta^i = \eta_\zeta^1$
  - $\Longrightarrow$ likely that η_ζ^i will wander off
    - where to?

A bit more on eta-tilde - Case I

- Suppose that

$$\zeta_t = \begin{cases} -1 \text{ with probability } \frac{1}{2} \\ +1 \text{ with probability } \frac{1}{2} \end{cases}$$

- My experience (not a theorem):

$$\begin{aligned} \lim_{i \longrightarrow \infty} \eta_\zeta^i &= \pi/2 \text{ for large enough initial value} \\ \lim_{i \longrightarrow \infty} \eta_\zeta^i &= 0 \text{ for low enough initial value} \end{aligned}$$

- Note that

$$\max_{\eta_\zeta} \left| \sin\left(\eta_\zeta \zeta_t\right) \right| = \pi/2$$

- Thus, impact sunspot is made as large as possible when $\eta_\zeta^i \longrightarrow \pi/2$.

A bit more on eta-tilde - Case II

- Suppose that

$$\zeta_t \sim N(0,1)$$

- Again it looks like convergence to different sunspot solutions is possible depending on initial conditions

- Much work remains to be done

References

- McGough, B., Q. Meng, and J. Xue, 2011, Indeterminacy and E-stability in real business cycle models with factor-generated externalities, manuscript.
  - Paper provides a nice example of a sunspot in a linearized RBC-type model that is learnable.
- Slides on Blanchard-Kahn conditions (& sunspots); available on line
- Slides on PEA; available on line