Models with Heterogeneous Agents
Introduction

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Overview

- "Simple" model with heterogeneous agents
  - understanding the complexity of these models
  - role of aggregate uncertainty
  - role of incomplete markets
- Solving the Aiyagari model
  - basic numerical techniques (refresher)
- Does heterogeneity matter?

Overview continued

- Avoiding complexity
  - heterogeneity only within the period
  - partial equilibrium
  - are two agents enough?
- Other models with heterogeneity
  - New Keynesian model
  - Multiplicity & domino effects due to tax externality
  - macro model with search frictions

First model with heterogeneous agents

- Agents are *ex ante* the same,
  but face different idiosyncratic shocks
  $\Longrightarrow$ agents are different *ex post*

- Incomplete markets
  $\Longrightarrow$ heterogeneity cannot be insured away

Individual agent

- Subject to employment shocks:
    - $\varepsilon_{i,t} \in \{0,1\}$
- Incomplete markets
    - only way to save is through holding capital
    - borrowing constraint $k_{i,t+1} \geq 0$

Aggregate shock

- $z_t \in \{z^b, z^g\}$
- z_t affects
  1. aggregate productivity
  2. probability of being employed

Laws of motion

- z_t can take on two values
- $\varepsilon_{i,t}$ can take on two values
- probability of being (un)employed depends on z_t
- transition probabilities are such that
  - unemployment rate only depends on current z_t
  - thus
    - $u_t = u^b$ if $z_t = z^b$ &
    - $u_t = u^g$ if $z_t = z^g$
    - with $u^b > u^g$.

Firm problem

$$
\begin{aligned}
r_t &= \alpha z_t K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t &= (1-\alpha)\, z_t K_t^{\alpha} L_t^{-\alpha}
\end{aligned}
$$

These are identical to those of the rep. agent version

Government

$$
\begin{aligned}
\tau_t w_t \bar{l}(1 - u(z_t)) &= \mu w_t u(z_t) \\
\tau_t &= \frac{\mu u(z_t)}{\bar{l}(1 - u(z_t))}
\end{aligned}
$$

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathrm{E} \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1-\tau_t) w_t \bar{l} \varepsilon_{i,t} + \mu w_t (1-\varepsilon_{i,t}) + (1-\delta) k_{i,t}$$
$$k_{i,t+1} \geq 0$$

- this is a relatively simple problem
  if processes for r_t and w_t are given

Individual agent - foc

$$
\begin{aligned}
\frac{1}{c_{i,t}} &\geq \beta \mathsf{E}_t \left[\frac{1}{c_{i,t+1}} (r_{t+1} + 1 - \delta) \right] \\
0 &= k_{i,t+1} \left(\frac{1}{c_{i,t}} - \beta \mathsf{E}_t \left[\frac{1}{c_{i,t+1}} (r_{t+1} + 1 - \delta) \right] \right) \\
c_{i,t} + k_{i,t+1} &= r_t k_{i,t} + (1 - \tau_t) w_t \bar{l} \varepsilon_{i,t} + \mu w_t (1 - \varepsilon_{i,t}) + (1 - \delta) k_{i,t} \\
k_{i,t+1} &\geq 0
\end{aligned}
$$

What aggregate info do agents care about?

- current **and** future values of r_t and w_t

- the period-t values of r_t and w_t
  - only depend on aggregate capital stock, K_t, & z_t
  - !!! In most models, prices also depend on other characteristics of the distribution

What aggregate info do agents care about?

- the future values, i.e., $r_{t+\tau}$ and $w_{t+\tau}$ with $\tau > 0$ depend on
  - future values of mean capital stock, i.e. $K_{t+\tau}$, & $z_{t+\tau}$
- $\Longrightarrow$ agents are interested in all information that forecasts K_t
- $\Longrightarrow$ typically this includes the complete cross-sectional distribution of employment status and capital levels (**even when** you only forecast futures means like you do here)

Equilibrium - first part

- Individual policy functions that solve agent's max problem
- A wage and a rental rate given by equations above.

Equilibrium - second part

- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

  - f_t = cross-sectional distribution of beginning-of-period capital and the employment status *after* the employment status has been realized.
  - z_{t+1} does *not* affect the cross-sectional distribution of capital
  - z_{t+1} does affect the *joint* cross-sectional distribution of capital and employment status

Transition law & timing

- f_t & $z_t \Longrightarrow f_t^{\text{end-of-period}}$
- $f_t^{\text{end-of-period}}$ & $z_{t+1} \Longrightarrow f_{t+1}^{\text{beginning-of-period}} \equiv f_{t+1}$

Transition law & timing

- Let g_t be the cross-sectional distribution of capital (so without any info on employment status)
- Why can I write
$$g_{t+1} = \Upsilon_g(z_t, f_t)?$$

Transition law & continuum of agents

$$
\begin{aligned}
g_{t+1} &= \Upsilon_g(z_t, f_t) \\
f_{t+1} &= \Upsilon(z_{t+1}, z_t, f_t)
\end{aligned}
$$

Why are these exact equations without additional noise?

- continuum of agents $\Longrightarrow$ rely on law of large numbers to average out idiosyncratic risk
- are we allowed to do this?

Recursive equilibrium?

Questions

1. Does an equilibrium exist?
   1. If yes, is it unique?
2. Does a recursive equilibrium exist?
   1. If yes, is it unique?
   2. If yes, what are the state variables?

Recursive equilibrium?

Jianjun Miao (JET, 2006): a recursive equilibrium exist for following state variables:

- usual set of state variables, namely
  - individual shock, $\varepsilon_{i,t}$
  - individual capital holdings, $k_{i,t}$
  - aggregate productivity, z_t
  - joint distribution of income and capital holdings, f_t
- and *cross-sectional distribution of expected payoffs*

Unique?

Heterogeneity $\Longrightarrow$ more reasons to expect multiplicity

- my actions depend on what I think others will do
- heterogeneity tends to go together with frictions and multiplicity more likely with frictions
    - e.g. market externalities

Wealth-recursive (WR) equilibrium

- WR equilibrium is a recursive equilibrium with only $\varepsilon_{i,t}$, $k_{i,t}$, z_t, and f_t as state variables.
  (Also referred to as Krusell-Smith (KS) recursive equilibrium)

- Not proven that WR equilbrium exists in model discussed here (at least not without making unverifiable assumptions such as equilibrium is unique for all possible initial conditions)

- Kubler & Schmedders (2002) give examples of equilibria that are *not* recursive in wealth
  (i.e., wealth distribution by itself is not sufficient)

Wealth distribution not sufficient - Example

- Static economy
  two agents, $i = 1, 2$, two goods, $j = A, B$
- Utility: $\ln q_A + \ln q_B$
- Endowments in state I: $\omega_{1,A} = \omega_{2,A} = 1; \omega_{1,B} = \omega_{2,B} = 1$
- Endowments in state II:
  $\omega_{1,A} = \omega_{2,A} = 1; \omega_{1,B} = \omega_{2,B} = 10/9$
- Normalization: $p_A = 1$

Wealth distribution not sufficient - Example

- State I:
  - equilibrium: $p_B = 1; q_{1,A} = q_{2,A} = 1; q_{1,B} = q_{2,B} = 1$
    wealth of each agent: $= 2$
- State II:
  - equilibrium: $p_B = 0.9; q_{1,A} = q_{2,A} = 1; q_{1,B} = q_{2,B} = 10/9$
    wealth of each agent: $= 2$
- Thus: same wealth levels, but different outcome

How to proceed?

- Wealth distribution may not be sufficient!
- For numerical analysis less problematic: It typically leaves stuff out
  After obtaining solution, you should check whether the approximation is accurate or not

How to proceed?

- For now we assume that a wealth recursive equilibrium exists (or an approximation based on it is accurate)
- This is still a tough numerical problem

If a wealth recursive equilibrium exists

- Suppose that recursive RE for usual state space exists
  - $s_{i,t} = \{\varepsilon_{i,t}, k_{i,t}, s_t\} = \{\varepsilon_{i,t}, k_{i,t}, z_t, f_t\}$
- Equilibrium:
  - $c(s_{i,t})$
  - $k(s_{i,t})$
  - $r(s_t)$
  - $w(s_t)$
  - $\Upsilon(z_{t+1}, z_t, f_t)$

Alternative representation state space

- Suppose that recursive RE for usual state space exist
  - $s_{i,t} = \{\varepsilon_{i,t}, k_{i,t}, s_t\} = \{\varepsilon_{i,t}, k_{i,t}, z_t, f_t\}$
- What determines current shape f_t?
  - z_t, z_{t-1}, f_{t-1} or
  - $z_t, z_{t-1}, z_{t-2}, f_{t-2}$ or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, f_{t-3}$ or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, z_{t-4}, f_{t-4}$ or
  - $\cdots$

No aggregate uncertainty

$$s_t = \lim_{n \longrightarrow \infty} \{z_t, z_{t-1}, \cdots, z_{t-n}, f_{t-n}\}$$

- Why is this useful from a numerical point of view?
  - when z_t is stochastic
  - when z_t is not stochastic (case of no aggregate uncertainty)

No aggregate uncertainty

State variables

$$\lim_{n \longrightarrow \infty} \{z_t, z_{t-1}, \cdots, z_{t-n}, f_{t-n}\}$$

- **If**
  1. $z_t = z \; \forall t$ and
  2. effect of initial distribution dies out
- **then** s_t constant
  - distribution still matters!
  - but it is no longer a *time-varying* argument

Aggregation

Statement:

The representative agent model is silly,
because there is no trade in this model,
while there is lots of trade in financial assets in reality

Aggregation

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The representative agent model is silly,
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while there is lots of trade in financial assets in reality

Problem with statement:

RA is justified by complete markets
which relies on lots of trade

Complete markets & exact aggregation

- economy with ex ante identical agents
- J different states
- complete markets $\Longrightarrow J$ contingent claims

Complete markets & exact aggregation

$$\max_{c_{i,t},b^1_{i,t+1},\cdots,b^J_{i,t+1}} \frac{(c_{i,t})^{1-\gamma}}{1-\gamma} + \beta \mathsf{E}_t \left[v(b^1_{i,t+1},\cdots,b^J_{i,t+1}) \right]$$

$$\text{s.t.} \quad c_{i,t} + \sum_{j=1}^{J} q^j b^j_{i,t+1} = y_{i,t} + \sum_{j=1}^{J} I(j^*) b^j_{i,t}$$

$$b^j_{i,t+1} > \overline{b} \;\; \text{with } \overline{b} < 0$$

Euler equations individual

$$q^j \left(c_{i,t}\right)^{-\gamma} = \beta \left(c^j_{i,t+1}\right)^{-\gamma} \text{prob}(j) \qquad \forall j$$

This can be written as follows:

$$c_{i,t} = \left(\frac{\beta \text{prob}(j)}{q^j}\right)^{-1/\gamma} c^j_{i,t+1} \qquad \forall j$$

Aggregation

Aggregation across individual i of

$$c_{i,t} = \left(\frac{\beta \text{prob}(j)}{q^j}\right)^{-1/\gamma} c^j_{i,+1} \quad \forall j$$

gives

$$C_t = \left(\frac{\beta \text{prob}(j)}{q^j}\right)^{-1/\gamma} C^j_{t+1} \quad \forall j,$$

which can be rewritten as

$$q^j \left(C_t\right)^{-\gamma} = \beta \left(C^j_{t+1}\right)^{-\gamma} \text{prob}(j) \quad \forall j$$

Use equilibrium condition

- In equilibrium:
  - aggregate consumption equals aggregate income or
  - contingent claims are in zero net supply
- Thus

$$q^j \left(Y_t\right)^{-\gamma} = \beta \left(Y^j_{t+1}\right)^{-\gamma} \text{prob}(j) \quad \forall j$$

Back to represenative agent model

- Idential FOCs come out of this RA model:

$$\max_{C,B^1_{+1},\cdots,B^J_{+1}} \frac{(C_t)^{1-\gamma}}{1-\gamma} + \beta \mathsf{E}_t \left[v(B^1_{t+1},\cdots,B^J_{t+1}) \right]$$

$$s.t. C_t + \sum_{j=1}^{J} q^j B^j_{t+1} = Y_t + \sum_{j=1}^{J} I(j^*) B^j_t$$

$$B^j_{t+1} > \bar{b} \text{ with } \bar{b} < 0$$

Back to model with heterogeneous agents

1. (For now) no aggregate risk
   Aiyagari model
2. We simplify the standard setup as follows:
   - Replace borrowing constraint by penalty function
     $\Longrightarrow$ going short is possible but costly
   - workers have productivity insteady of unemployment shocks
     $\varepsilon_{i,t}$ with $\mathrm{E}[\varepsilon_{i,t}] = 1$

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathsf{E} \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t}) - \frac{\zeta_1}{\zeta_0} \exp(-\zeta_0 k_{i,t}) - \zeta_2 k_{i,t}$$

s.t.

$$c_{i,t} + k_{i,t} = r_t k_{i,t-1} + w_t \varepsilon_{i,t} + (1-\delta) k_{i,t-1}$$

First-order condition

$$-\frac{1}{c_{i,t}} + \zeta_1 \exp(-\zeta_0 k_{i,t}) - \zeta_2 + \mathsf{E}_t \left[\frac{\beta}{c_{i,t+1}} \left(r_{t+1} + 1 - \delta \right) \right] = 0$$

Penalty function

- advantage of ζ_2 term:
  - supppose $\bar{k}$ and $\bar{r}$ are steady states of rep agent model
  - if
  $$\zeta_2 = \zeta_1 \exp(-\zeta_0 \bar{k})$$
  then steady state of this model is same

Equilibrium

- Unit mass of workers, $L_t = 1$
- Competitive firm $\Longrightarrow$ agent faces competitive prices
  - $w_t = (1-\alpha) K_t^{\alpha} L_t^{1-\alpha} = (1-\alpha) K_t^{\alpha}$
  - $r_t = \alpha K_t^{\alpha-1} L_t^{\alpha} = \alpha K_t^{\alpha-1}$
- No aggregate risk so

$$K_t = K$$

- How to find the equilibrium K?

Algorithm

- Guess a value for r
- This implies values for K^{demand} and w
- Solve the individual problem with these values for r & w
- Simulate economy & calculate the supply of capital, K^{supply}
- If $K^{\text{supply}} < K^{\text{demand}}$ then r too low so raise r, say

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

- Iterate until convergence

Algorithm

Using

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

to solve

$$K^{\text{demand}}(r) = K^{\text{supply}}(r)$$

not very efficient

- Value of λ may have to be very low
- More efficient to use equation solver to solve fro r

Use Dynare to solve indiv. policy rule

- Specify guess for r in mother Matlab file
- Make r parameter in *.mod file
- In mother Matlab file write r using

```
save r_file r
```

Use Dynare to solve indiv. policy rule

- In *.mod file use

```
load r_file
set_param_value('r',r)
```

instead of

```
r = 0.013;
```

Simulate yourself using Dynare solution

1. Use values stored by Dynare or
2. Replace Dynare's disp_dr.m with my alternative

- this saves the policy functions *exactly as shown on the screen*
  - asa matrix
  - in a Matlab data file `dynarerocks.mat`
  - under the name `decision`

Does heterogeneity matter?

- Important to distinguish between
  - (i) theoretical results
  - (ii) their quantitative importance
- Examples
  - no aggregation in presence of incomplete markets
  - Arrow's impossibility theorem

Does incompleteness/heterogeneity matter?

- Take model with
  - infinitely-lived agents
  - no complete markets
    - e.g. agents can only borrow/lend through a safe asset
- $\Longrightarrow$ no aggregation to RA model possible
- But in many models effects small
  - why does infinitely-lived agent assumption matter?

Does incompleteness/heterogeneity matter?

- Effects often small for
  - asset prices
  - aggregate series
    - except possibly some impact on means
- Effects much bigger for
  - individual series, e.g. $VAR(c_{i,t}) >> VAR(C_t)$

Avoiding complexity

- heterogeneity only within the period
- partial equilibrium
- two agents?

Avoiding complexity

- Lesson learned above:
    - incomplete asset markets don't do much in many environments
- This implies you should
    - either use more interesting environment
    - or use complete asset markets
- This does *NOT* imply you should eliminate heterogeneity from your models

Only heterogeneity within period

- Household with heterogeneous members within the period:
  - members are on their own and face frictions. E.g.
    - cannot transfer funds to each other
    - cannot transfer information
- At the end of period:
  - all members bring this period's revenues to household who makes savings decision

Partial equilibrium

Which of the following two would you prefer?

- **General equilibrium** asset pricing model that generates *unrealistic* asset prices
- **Partial equilibrium** model that uses *realistic* asset prices as exogenous processes

Partial of general equilibrium?

What about follwing example

- Government sets interest rates
  - !!! Government cannot set current r_t nor r_{t+1}.
  - Suppose it sets $\mathsf{E}_t\left[r_{t+1}\right]$. E.g.,

$$\mathsf{E}_t\left[r_{t+1}\right] = (1-\rho_r)r^* + \rho_r r_t + \varepsilon_{r,t}$$

- Government supplies capital to implement this.

Partial of general equilibrium?

- These expenditures are financed by lump sum taxes.
- State variables are
  - $k_{i,t}$
  - $\varepsilon_{i,t}$
  - z_t
  - K_t but no higher-order moments
  - $\mathsf{E}_t\left[r_{t+1}\right]$ or ???

Small number of agents

Consider following endowment economy

- Type 1 agent receives $z_{1,t}$
- Type 2 agent receives $z_{2,t}$
- average endowment z_t

$$z_t = 0.5 z_{1,t} + 0.5 z_{2,t}$$

- agents smooth idiosyncratic risk by trading in safe bonds

Small number of agents

$$c_{i,t}^{-\gamma} \geq q_t \beta \mathsf{E}_t \left[c_{i,t+1}^{-\gamma} \right]$$

$$\left(b_{i,t+1} - \bar{b} \right) \left(c_{i,t}^{-\gamma} - q_t \beta \mathsf{E}_t \left[c_{i,t+1}^{-\gamma} \right] \right) = 0$$

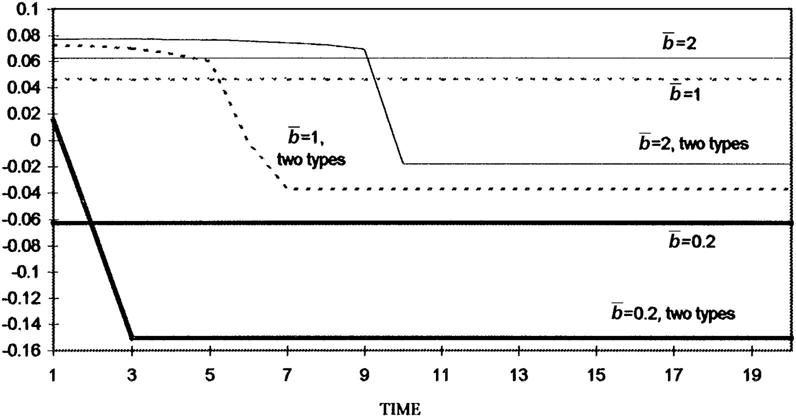
$$b_{i,t+1} \geq \bar{b}$$

$$b_{1,t+1} + b_{2,t+1} = 0$$

Idiosyncratic risk

- You want to study effect of idiosyncratic risk.
- Suppose agent 1 repeatedly gets the bad shock
- Difference with model with lots of types?
  - here: *lots* of agents always get *same* shock at *same* time
  - so what?

Idiosyncratic risk and interest rate



Heterogeneity in other models

- Standard New Keynesian model
- Simple static model with tax externality
- Standard model with search friction
  - multiple steady states
  - multiple solutions

New Keynesian models

- Calvo devil $\Longrightarrow$ heterogeneous price dispersion
- Standard approach:
  - only focus on *aggregates*
  - focus on linearized solution

Disaggregate results in NK models

- Suppose
  - all firms start with same price (for simplicity)
  - consider monetary tightening
- Aggregate:
  - downturn because of sticky prices

Disaggregate results in NK models

- Firms that are *not* constrained by Calvo devil: $p_i \downarrow$
    - their aggregate demand $\uparrow$ because $p_i/P \downarrow$
    - their aggregate demand $\downarrow$ because aggregate demand $\downarrow$
    - total effect can easily be $\uparrow$
- But empirical evidence suggests decline across different sectors

Asymmetry in New Keynesian models

Suppose goods are perfect substitutes

- Monetary tightening:
  - firms that are *not* constrained by Calvo devil: $p_i \downarrow$
  - $\Longrightarrow$ firms constrained by Calvo devil sell 0
  - $\Longrightarrow$ same outcome as fully flexible case
  - $\Longrightarrow \Delta Y = 0$

Asymmetry in New Keynesian models

Suppose goods are perfect substitutes

- Monetary stimulus:
  - Firms that are constrained by Calvo devil: $\Delta p_i = 0$
  - $\Longrightarrow$ firms *not* constrained by Calvo devil: $\Delta p_i = 0$
  - $\Longrightarrow$ same outcome as fixed P case
  - $\Longrightarrow \Delta Y < 0$

The true New Keynesian models

Conclusion:

True New Keynesian models are much more interesting than the linearized version the profession is obsessed with

Is the true NK model also more realistic?

Tax externality

- Static model
- N different skill levels
  - z_k, $k = 1, \cdots, N$
  - $z_1 = \bar{z}$
  - $z_{k+1} = z_k + \varepsilon$
- unemployed get benefits

Tax externality

animated picture

Search model

Consider the following model

- unit mass of workers
- workers need to search to find a job
- employers post vacancy to find worker
- productivity of matched pairs distributed i.i.d
    - so each period a new draw

Key decision

- Given value of $\varepsilon_{i,t}$ is it better to
  1. produce or
  2. quit and enjoy leisure?

Equation for cut-off value

- The cut-off value $\bar{\varepsilon}_i$ given by

$$0 = \bar{\varepsilon}_i + G - b - W$$

  - G : continuation value of ending period in match
    - does not depend on $\varepsilon_{i,t}$ (i.i.d. assumption)
    - does depend on $\bar{\varepsilon}_i$
  - W : continuation value of ending period not in match
    - also depends on $\bar{\varepsilon}_i$

Solution for cut-off value

- We are looking for a solution to

$$0 = \bar{\varepsilon}_i + G(\bar{\varepsilon}_i) - b - W(\bar{\varepsilon}_i)$$

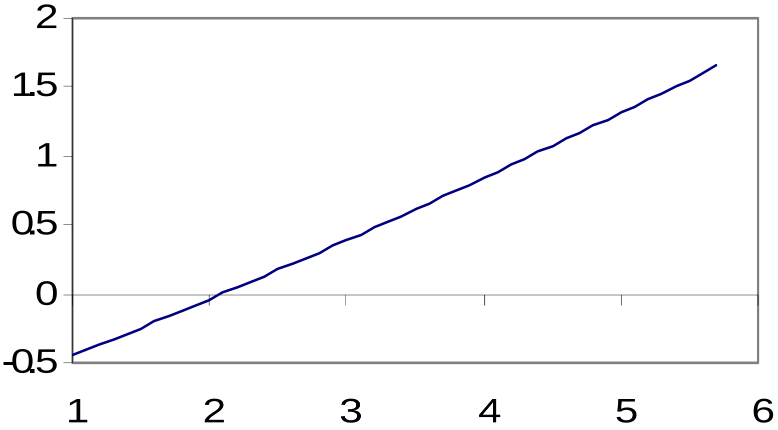
- Unique solution if

$$\frac{\partial \left(\bar{\varepsilon}_i + G(\bar{\varepsilon}_i) - b - W(\bar{\varepsilon}_i)\right)}{\partial \bar{\varepsilon}_i} > 0 \quad \forall \bar{\varepsilon}_i$$

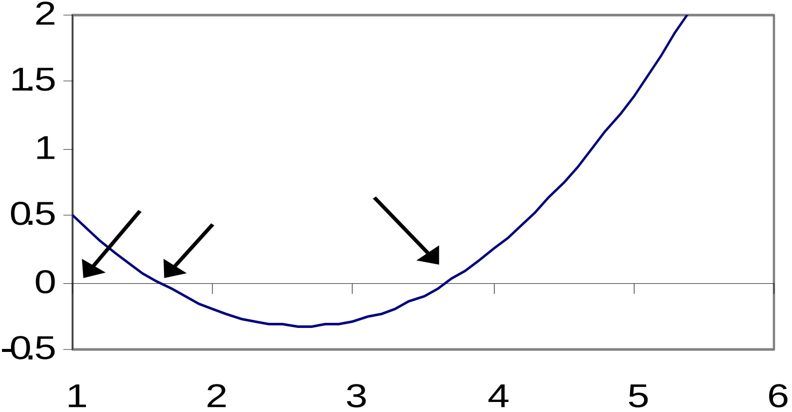
  - but typically we have

$$\frac{\partial \left(G(\bar{\varepsilon}_i) - W(\bar{\varepsilon}_i)\right)}{\partial \bar{\varepsilon}_i} < 0 \quad \text{for some } \bar{\varepsilon}_i$$

Unique steady state



Multiple steady state case



Reasons for multiplicity

- *expectations* about the stability of future matches
  - as in example above
- market activity could affect revenues
  - as in static example with tax externality

Tax externality and multiplicity

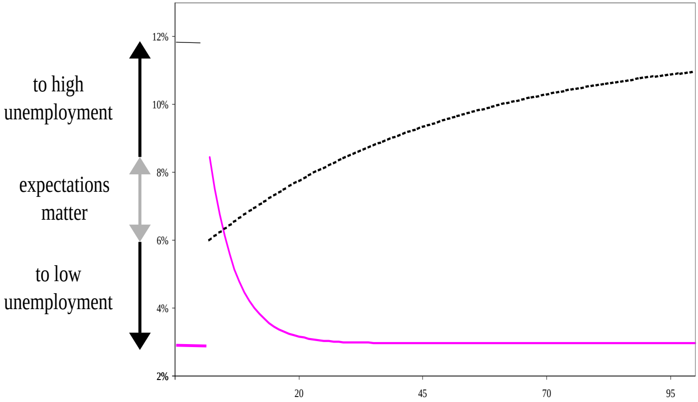
Easy to get two steady states

- Low (high) taxes $\Longrightarrow$
- Surplus high (low) $\Longrightarrow$
- Job destruction low (high) $\Longrightarrow$
- Unemployment rate low (high) $\Longrightarrow$
- Taxes indeed low (high)

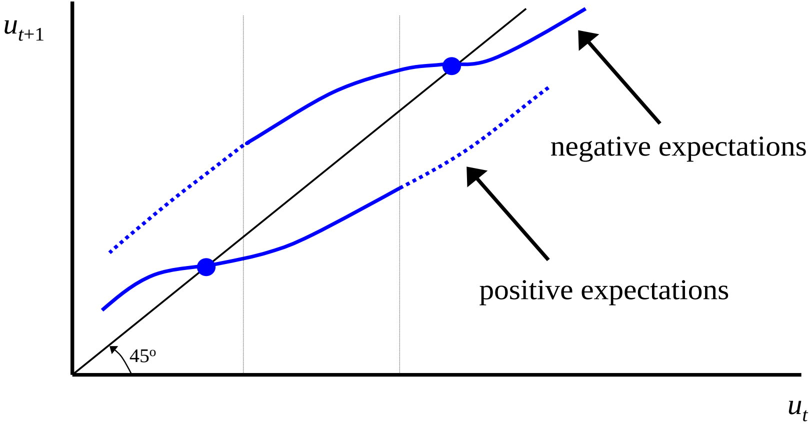
Multiple what?

Multiple steady states $\not\Rightarrow$ multiple solutions

Transition dynamics I



Transition dynamics II



Why is it hard to get this published in AER?

- What aspect of distribution determines whether this is quantitatively important?
- How do you get data on this?

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