Solving Models with Heterogeneous Expectations

Wouter J. Den Haan
London School of Economics

August 29, 2014

Overview

1. Current approaches to model heterogeneous expectations
2. Numerical algorithm to solve models with rational and irrational agents the right way

Modelling heterogeneous expectations

- Agent-based modelling:
  - several papers have "fundamental" or "rational" agents but terminology is very misleading
  - lack of forward looking agents makes numerical analysis very straightforward (just simulate the economy)
- Rule of thumb and rational agents
  - popular in model with New Keynesian Phillips curve
  - But NK Phillips curve has been derived under assumption of homogeneous expectations

Some papers do combine both elements

Nice examples in the literature:

- Nunes (*Macroeconomic Dynamics*, 2009)
  - rational agents and agents that learn
- Molnar (2010)
  - rational agents and backward looking agents
  - fractions of each endogenous

Nunes (2009)

NK Phillips curve:

$$\pi_t = \kappa z_t + \beta \widetilde{\mathsf{E}}_t \pi_{t+1}$$

IS curve:

$$z_t = \widetilde{\mathsf{E}}_t z_{t+1} - \sigma^{-1} \left(r_t - r_t^n - \widetilde{\mathsf{E}}_t \pi_{t+1} \right)$$

Nunes (2009)

Natural rate:

$$r_t^n = \rho r_{t-1}^n + \varepsilon_t$$

Policy:

$$r_t = \pi^* + \phi_\pi \left(\pi_t - \pi^*\right) + \phi_z z_t$$

Nunes (2009)

Nice:

- Rational agents are truly rational

Not nice:

- Nunes follows the standard approach:
  1. use a representative agent model
  2. simply replace the expectation by a weighted average of agents expectations

But how do I know these are the right relationships if agents have heterogeneous expectations

Molnar (2010)

Model:

$$\begin{aligned} p_t &= \lambda \mathsf{E}_t\left[p_{t+1}\right] + m_t \\ m_t &= \rho m_{t-1} + \varepsilon_t, \quad \rho \in [0,1) \end{aligned}$$

Again, $\mathsf{E}_t\left[p_{t+1}\right]$ is a weighted average of the expectations of different types

New Keynesian model and aggregation

NK Phillips curve:

$$\pi_t = \beta \mathsf{E}_t \left[\pi_{t+1}\right] + \lambda y_t$$

- Question: Does this equation hold in models with heterogeneous agents with $\mathsf{E}_t \left[\pi_{t+1}\right]$ replaced by weighted average?
- Branch and McGough (2009):
  - there is a set of assumptions for which the answer is yes
  - assumptions are restrictive
  - even necessary conditions are shown to be restrictive

Alternative setup

- Model behavior of *individual* agents from the ground up
  - some rational
  - some not rational
- Explicit aggregate their behavior to get aggregate behavior
- Can we solve these models? Yes
  - using the tools learned in this course

Environment

- unit mass of firms
- half has rational expectations
- half has "type A" expectations
- for now fractions are fixed

Firm output

All firms have the same production function

$$y_i = z_i \left(z_1 n_{1,-1}^{\alpha} + z_2 n_{2,-1}^{\alpha} \right)$$

- two production processes
- n_1 and n_2 are chosen in previous period
- z_i : idiosyncratic shock
- z_1 and z_2 : aggregate (common) shocks

Exogenous random processes

$$
\begin{aligned}
z_i &= (1-\rho_i) + \rho_i z_{i,-1} + e_i, \quad e_i \sim N(0,\sigma_i^2) \\
z_1 &= (1-\rho) + \rho z_{1,-1} + e_1, \quad e_1 \sim N(0,\sigma^2) \\
z_2 &= (1-\rho) + \rho z_{2,-1} + e_2, \quad e_2 \sim N(0,\sigma^2)
\end{aligned}
$$

Problem rational agent

$$
\begin{array}{c}
v(n_{1,-1}, n_{2,-1}, z) \\
= \\
\max_{n_1, n_2} \left(\begin{array}{c}
z_i \left(z_1 n_{1,-1}^{\alpha} + z_2 n_{2,-1}^{\alpha} \right) - w(N_{-1}) - 0.5\eta (n - n_{-1})^2 \\
+\beta \mathsf{E}_t \left[v(n_1, n_2, z_{+1}) \right]
\end{array} \right)
\end{array}
$$

where

$$
\begin{array}{rcl}
n & = & n_1 + n_2, \\
N & : & \text{aggregate employment}
\end{array}
$$

FOCs rational firm

$$\alpha\beta \mathsf{E}\left[z_{i,+1}\right]\mathsf{E}\left[z_{1,+1}\right]n_1^{\alpha-1}-\beta\mathsf{E}\left[w_{+1}\right]$$
$$-\eta\left(n-n_{-1}\right)+\eta\beta\mathsf{E}\left[n_{+1}-n\right]=0$$

$$\alpha\beta \mathsf{E}\left[z_{i,+1}\right]\mathsf{E}\left[z_{2,+1}\right]n_2^{\alpha-1}-\beta\mathsf{E}\left[w_{+1}\right]$$
$$-\eta\left(n-n_{-1}\right)+\eta\beta\mathsf{E}\left[n_{+1}-n\right]=0$$

FOCs type A firm

$$\alpha\beta\widehat{\mathsf{E}}\left[\hat{z}_{i,+1}\right]\widehat{\mathsf{E}}\left[z_{1,+1}\right]\hat{n}_1^{\alpha-1} - \beta\widehat{\mathsf{E}}\left[w_{+1}\right]$$
$$-\eta\left(\hat{n}-\hat{n}_{-1}\right)+\eta\beta\widehat{\mathsf{E}}\left[\hat{n}_{+1}-\hat{n}\right]=0$$

$$\alpha\beta\widehat{\mathsf{E}}\left[\hat{z}_{i,+1}\right]\widehat{\mathsf{E}}\left[z_{2,+1}\right]\hat{n}_2^{\alpha-1} - \beta\widehat{\mathsf{E}}\left[w_{+1}\right]$$
$$-\eta\left(\hat{n}-\hat{n}_{-1}\right)+\eta\beta\widehat{\mathsf{E}}\left[\hat{n}_{+1}-\hat{n}\right]=0$$

Labor supply

$$w = \omega_0 + \omega_1 \left(N_{-1} - 1 \right)$$

$$N = \frac{\int n_i di + \int \hat{n}_i di}{2}$$

Normalization

$$\omega_0 = \alpha 0.5^{\alpha - 1}$$

Steady state values:

$$\begin{aligned} n_1 &= n_2 = \hat{n}_1 = \hat{n}_2 = 0.5 \\ w &= \omega_0 \\ N &= 1 \end{aligned}$$

Solving the rational firm problem

- Rational firm's problem easy if I know ....

Solving the rational firm problem

- Rational firm's problem easy if I know ....
- dgp for wages (or N)

Problem for type A

Just as easy or easier

Solving the complete problem

- Use KS iteration scheme

Solving the complete problem

- Use KS iteration scheme
- PEA

Varying fractions of types

- state-dependent switching
- probability of switching depends on profitability
- E.g.

$$
\begin{aligned}
P_{AR} &= \frac{1-\rho_p}{2} + \rho_p P_{AR,-1} + \eta \left(F_R - F_A \right) \\
P_{RA} &= \frac{1-\rho_p}{2} + \rho_p P_{RA,-1} - \eta \left(F_R - F_A \right)
\end{aligned}
$$

where

F_R : average profits made by rational firms

F_A : average profits made by type A firms

Implications for algorithms?

- What are the implications for deriving ALM?
- What are the implications for individual firm problem?

What is wrong?

```
alpha*n_1_R^(alpha-1)*z_i(+1)*z_1(+1)
=
wage_exp_R
+eta*(n_R-n_R(-1)
```

$$-(1-\texttt{prob}) \quad *\texttt{eta}*(\texttt{n\_R}(+1)-\texttt{n\_R})$$
$$-\texttt{prob} \quad \texttt{eta}*(\texttt{n\_A}(+1)-\texttt{n\_R})$$

How to do it correctly with Dynare?

```
alpha*n_1_R^(alpha-1)*z_i(+1)*z_1(+1)
=
wage_exp_R
+eta*(n_R-n_R(-1)
```

$$-\begin{pmatrix} 1 - \text{prob} \\ \text{prob} \end{pmatrix} \begin{pmatrix} *\text{eta} * (\text{n\_R}(+1) - \text{n\_R}) \\ \text{eta} * (??? - \text{n\_R}) \end{pmatrix}$$

How to do it correctly with Dynare?

- Suppose `???` in the previous slide is an *endogenous* rule that I have to solve for.
  Can I solve for the policy rule of `???` and `n_R` with Dynare?

How to do it correctly with Dynare?

- Suppose `???` in the previous slide is an *endogenous* rule that I have to solve for.
  Can I solve for the policy rule of `???` and `n_R` with Dynare?

- Yes, but you need two programs and iterate back and forth

How to do it correctly with Dynare?

- Can I do this with *first-order* perturbation?

How to do it correctly with Dynare?

- Can I do this with *first-order* perturbation?
- No, because of quadratic cost

Switching types and valuation problems

example above:

- rational firm does think through implications of choices for his irrational self
- But there is no "valuation" problem to assess value of irrational self
- Is that problematic here?
  No, a rational firm just needs to know how an irrational firm chooses employment

Switching types and valuation problems

new example

- simpler environment
    - no idiosyncratic shocks
- different question, namely:
    - calculate firm value, while rationally taking into account that you could become irrational
    - fixed probability of switching

Firm problem

$$v(n_{-1},z) = \max_n \begin{array}{c} zn_{-1}^{\alpha} - wn_{-1} + 0.5\eta(n-n_{-1})^2 \\ +\beta \mathsf{E}_t \left[\begin{array}{c} (1-\rho)\, v(n,z_{+1}) \\ \rho w(n,z_{+1}) \end{array} \right] \end{array}$$

$$= \begin{array}{c} zn_{-1}^{\alpha} - wn_{-1} + 0.5\eta(n^*-n_{-1})^2 \\ +\beta \mathsf{E}_t \left[\begin{array}{c} (1-\rho)\, v(n^*,z_{+1}) \\ \rho w(n^*,z_{+1}) \end{array} \right] \end{array}$$

Switching types and valuation problems

- $v(n_{-1}, z)$: value of a rational firm according to rational agent
- $w(n_{-1}, z)$: value of an irrational firm according to a rational agent, i.e. *using rational expectations*

Rational value of irrational firm

$$w(n_{-1},z) = \begin{array}{c} zn_{-1}^{\alpha} - wn_{-1} + 0.5\eta(\hat{n} - n_{-1})^2 \\ \\ +\beta \mathsf{E}_t \left[\begin{array}{c} (1-\rho)\, v(\hat{n}, z_{+1}) \\ \rho w(\hat{n}, z_{+1}) \end{array} \right] \end{array}$$

So again a standard problem

Learning problem

- Go back to original problem
  - but no idiosyncratic shocks
- Type A firms forecast using "least-squares" learning
- They use past observations to fit forecasting rule for
- $\widehat{\mathsf{E}}\left[z_1\right]$, $\widehat{\mathsf{E}}\left[z_2\right]$, $\widehat{\mathsf{E}}\left[w_{+1}\right]$, $\widehat{\mathsf{E}}\left[\hat{n}_{+1}\right]$

Forecasting rules

Example of forecasting rules

$$\widehat{\mathsf{E}}\left[y\right] = \hat{\omega}_0 + \hat{\omega}_1 y_{-1}$$

and $\hat{\omega}_0$ and $\hat{\omega}_1$ estimated using past observations

Weighted least-squares

$$y_t = bx_t + u_t$$

$$\begin{aligned}
\hat{b}_T &= \frac{\sum_{t=1}^{T} \beta^{T-t} x_t y_t}{\sum_{t=1}^{T} \beta^{T-t} x_t^2} \\
&= \frac{\sum_{t=1}^{T} \beta^{T-t} x_t y_t}{K_T}
\end{aligned}$$

Recursive weighted least-squares

- For the problem to be tractable we need recursive formulation for the forecasts made by type A firms
  - like in the Kalman filter

Recursive weighted least-squares

$$
\begin{aligned}
\hat{b}_{T+1} &= \frac{\sum_{t=1}^{T+1} \beta^{T+1-t} x_t y_t}{K_{T+1}} \\
&= \frac{\beta \sum_{t=1}^{T} \beta^{T-t} x_t y_t + x_{T+1} y_{T+1}}{K_{T+1}} \\
&= \frac{\beta K_T}{K_{T+1}} \hat{b}_T + \frac{x_{T+1} y_{T+1}}{K_{T+1}} \\
K_{T+1} &= \beta K_T + x_{T+1}^2
\end{aligned}
$$

FOCs rational firm

$$\alpha\beta\mathsf{E}\left[z_{1,+1}\right]n_1^{\alpha-1}-\beta\mathsf{E}\left[w_{+1}\right]$$
$$-\eta\left(n-n_{-1}\right)+\eta\beta\mathsf{E}\left[n_{+1}-n\right]=0$$

$$\alpha\beta\mathsf{E}\left[z_{2,+1}\right]n_2^{\alpha-1}-\beta\mathsf{E}\left[w_{+1}\right]$$
$$-\eta\left(n-n_{-1}\right)+\eta\beta\mathsf{E}\left[n_{+1}-n\right]=0$$

Algorithm

- Parameterize expectations
  - which one(s)?
- Simulate economy
- Update expectation

State variables for rational agent

???

Tough problem?

- This is a standard problem
- Many state variables?
  - possibly if type A agents forecast a lot of variables
  - but maybe you don't need all as state variables to get an accurate solution

References

- Branch, W.A., and B.McGough, 2009, A New Keynesian Model with Heterogeneous Expectations, Journal of Economic Dynamics and Control.
- Molnar, K., 2010, Learning with Expert Advice, Journal of the European Economic Association.
- Nunes, R., 2009, Learning the inflation target, Macroeconomic Dynamics.