Function Approximation

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Goal

Obtain an approximation for

$$f(x)$$

when

- $f(x)$ is unknown, but we have some information, or
- $f(x)$ is known, but too complex to work with

Information available

- **Either** finite set of derivatives
  - usually at one point
- **or** finite set of function values
  - $f_1, \cdots, f_m$ at m nodes, $x_1, \cdots, x_m$

Classes of approximating functions

1. polynomials
   - this still gives lots of flexibility
   - examples of second-order polynomials
     - $a_0 + a_1 x + a_2 x^2$
     - $a_0 + a_1 \ln(x) + a_2 \left(\ln(x)\right)^2$
     - $\exp\left(a_0 + a_1 \ln(x) + a_2 \left(\ln(x)\right)^2\right)$
2. splines, e.g., linear interpolation

Classes of approximating functions

- Polynomials and splines can be expressed as

$$f(x) \approx \sum_{i=0}^{n} \alpha_i T_i(x)$$

- $T_i(x)$: the *basis functions* that define the *class* of functions used, e.g., for regular polynomials:

$$T_i(x) = x^i.$$

- α_i : the coefficients that pin down the particular approximation

Reducing the dimensionality

unknown $f(x)$: infinite dimensional object

$\sum_{i=0}^{n} \alpha_i T_i(x)$: $n+1$ elements

General procedure

- Fix the order of the approximation n
- Find the coefficients $\alpha_0, \cdots, \alpha_n$
- Evaluate the approximation
- If necessary, increase n to get a better approximation

Weierstrass (sloppy definition but true)

Let $f : [a,b] \longrightarrow \mathbb{R}$ be any real-valued function. For large enough n, it is approximated arbitrarily well with the polynomial

$$\sum_{i=0}^{n} \alpha_i x^i.$$

Thus, we can get an accurate approximation if

- f is not a polynomial
- f is discontinuous

How can this be true?

How to find the coefficients of the approximating polynomial?

- With derivatives:
  - use the Taylor expansion
- With a set of points (nodes), $x_0, \cdots, x_m$, and function values, $f_0, \cdots, f_m$?
  - use projection
  - Lagrange way of writing the polynomial (see last part of slides)

Function fitting as a projection

Let

$$Y = \begin{bmatrix} f_0 \\ \vdots \\ f_m \end{bmatrix}, X = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \cdots & T_n(x_0) \\ T_0(x_1) & T_1(x_1) & \cdots & T_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_m) & T_1(x_m) & \cdots & T_n(x_m) \end{bmatrix}$$

then

$$Y \approx X\alpha$$

- We need $m \geq n+1$. Is $m = n+1$ as bad as it is in empirical work?
- What problem do you run into if n increases?

Orthogonal polynomials

- Construct basis functions so that they are orthogonal to each other, i.e.,

$$\int_a^b T_i(x)T_j(x)w(x)dx = 0 \quad \forall i,j \ni i \neq j$$

- This requires a particular weighting function (density), $w(x)$, and range on which variables are defined, $[a,b]$

Chebyshev orthogonal polynomials

- $$[a, b] = [-1, 1] \text{ and } w(x) = \frac{1}{(1 - x^2)^{1/2}}$$

- What if function of interest is not defined on $[-1, 1]$?

Constructing Chebyshev polynomials

- The basis functions of the Chebyshev polynomials are given by

$$\begin{aligned} T_0^c(x) &= 1 \\ T_1^c(x) &= x \\ T_{i+1}^c(x) &= 2xT_i^c(x) - T_{i-1}^c(x) \quad i > 1 \end{aligned}$$

Chebyshev versus regular polynomials

- Chebyshev polynomials, i.e.,

$$f(x) \approx \sum_{j=0}^{n} a_j T_j^c(x),$$

can be rewritten as regular polynomials, i.e.,

$$f(x) \approx \sum_{j=0}^{n} b_j x^j,$$

Chebyshev nodes

- The $n^{\text{th}}-$order Chebyshev basis function has n solutions to

$$T_n^c(x) = 0$$

- These are the n Chebyshev nodes

Discrete orthogonality property

- Evaluated at the Chebyshev nodes, the Chebyshev polynomials satisfy:

$$\sum_{i=1}^{n} T_j^c(x_i) T_k^c(x_i) = 0 \text{ for } j \neq k$$

- Thus, if

$$X = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \cdots & T_n(x_0) \\ T_0(x_1) & T_1(x_1) & \cdots & T_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_m) & T_1(x_m) & \cdots & T_n(x_m) \end{bmatrix}$$

then $X'X$ is a diagonal matrix

Uniform convergence

- Weierstrass $\Longrightarrow$ there is a good polynomial approximation
- Weierstrass $\not\Rightarrow f(x) = \lim_{n\to\infty} p_n(x)$ for every sequence $p_n(x)$
- If polynomials are fitted on Chebyshev nodes$\Longrightarrow$ even *uniform* convergence is guaranteed

Splines

Inputs:

1. $n+1$ nodes, $x_0, \cdots, x_n$
2. $n+1$ function values, $f(x_0) \cdots, f(x_n)$

- nodes are fixed $\Longrightarrow$ the $n+1$ function values are the *coefficients* of the spline

Piece-wise linear

- For $x \in [x_i, x_{i+1}]$

$$f(x) \approx \left(1 - \frac{x - x_i}{x_{i+1} - x_i}\right) f_i + \left(\frac{x - x_i}{x_{i+1} - x_i}\right) f_{i+1}.$$

- That is, a separate linear function is fitted on the n intervals
- Still it is easier/better to think of the coefficients of the approximating function as the $n + 1$ function values

Piece-wise linear versus polynomial

- Advantage: Shape preserving
  - in particular monotonicity & concavity (strict?)
- Disadvantage: not differentiable

Extra material

1. Lagrange interpolation
2. Higher dimensional polynomials
3. Higher-order splines

Lagrange interpolation

Let

$$L_i(x) = \frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$ and

$$f(x) \approx f_0 L_0(x) + \cdots + f_n L_n(x).$$

- Right-hand side is an n^{th}-order polynomial
- By construction perfect fit at the $n + 1$ nodes?
- $\Longrightarrow$ the RHS is the n^{th}-order approximation

Higher-dimensional functions

- second-order *complete* polynomial in x and y:

$$\sum_{0 \leq i+j \leq 2} a_{i,j} x^i y^j$$

- second-order *tensor product* polynomial in x and y:

$$\sum_{i=0}^{2} \sum_{j=0}^{2} a_{i,j} x^i y^j$$

Complete versus tensor product

- tensor product can make programming easier
  - simple double loop instead of condition on sum
- n^{th} tensor has higher order term than $(n+1)^{\text{th}}$ complete
  - 2^{nd}-order tensor has fourth-order power
  - at least locally, lower-order powers are more important
    $\Longrightarrow$ complete polynomial may be more efficient

Higher-order spline

Cubic (for example)

- !!! Same inputs as with linear spline, i.e. $n+1$ function values at $n+1$ nodes which can still be thought of as the $n+1$ coefficients that determine approximating function
- Now fit 3rd-order polynomials on each of the n intervals

$$f(x) \approx a_i + b_i x + c_i x^2 + d_i x^3 \text{ for } x \in [x_{i-1}, x_i].$$

What conditions can we use to pin down these coefficients?

Cubic spline conditions: levels

- We have $2+2(n-1)$ conditions to ensure that the function values correspond to the given function values at the nodes.
  - For the intermediate nodes we need that the cubic approximations of both adjacent segments give the correct answer. For example, we need that

$$\begin{aligned} f_1 &= a_1 + b_1 x_1 + c_1 x_1^2 + d_1 x_1^3 \text{ and} \\ f_1 &= a_2 + b_2 x_1 + c_2 x_1^2 + d_2 x_1^3 \end{aligned}$$

  - For the two endpoints, x_0 and x_{n+1}, we only have one cubic that has to fit it correctly.

Cubic spline conditions: 1st-order derivatives

- To ensure differentiability at the intermediate nodes we need

  $b_i + 2c_i x_i + 3d_i x_i^2 = b_{i+1} + 2c_{i+1} x_i + 3d_{i+1} x_i^2$ for $x_i \in \{x_1, \cdots, x_{n-}$

  which gives us $n - 1$ conditions.

Cubic spline conditions: 2^{nd}-order derivatives

- To ensure that second derivatives are equal we need

$$2c_i + 6d_i x_i = 2c_{i+1} + 6d_{i+1} x_i \text{ for } x_i \in \{x_1, \cdots, x_{n-1}\}.$$

- We now have $2 + 4(n-1) = 4n - 2$ conditions to find $4n$ unknowns.
- We need two additional conditions; e.g. that 2^{nd}-order derivatives at end points are zero.

Splines - additional issues

- (standard) higher-order splines do not preserve shape
- higher-order difficult for multi-dimensional problems
- first-order trivial for multi-dimensional problems
    - if interval is small then nondifferentiability often doesn't matter

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