Value Function Iteration
versus
Euler equation methods

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Overview

1. How to do value function iteration (VFI)
2. VFI versus Euler equation methods
   1. convergence
   2. speed
   3. complex problems
Bellman equation

\[ V(x) = \max_{x_{+1} \in \Gamma(x)} U(x, x_{+1}) + E_t [\beta V(x_{+1})] \]
Essence of VFI

- $V^i(x)$: flexible functional form
  - piecewise linear (or higher-order spline)
  - discrete valued function (if $\Gamma(x)$ has $\chi < \infty$ elements)
  - quadratic (or higher-order polynomial)

- $V^{i+1}(x)$ is obtained from

$$V^{i+1}(x) = \max_{x+1 \in \Gamma(x)} U(x, x+1) + E_t \left[ \beta V^i(x+1) \right]$$
Essence of VFI

- This works in general

- However, on a computer the functional form of $V^i(x)$ must stay the same
  (so computer can store coefficients characterizing function)
Possible ways to implement VFI

1. Linear-Quadratic
   - $U(\cdot)$ is quadratic and constraints are linear
     $\implies V^i(\cdot)$ would remain quadratic
   - !!! To get a true first-order approximation to policy function
     you cannot take linear approximation of constraints
     $\implies$ either get rid of constraint by substitution or use the
     "correct" LQ approximation (see perturbation slides)

2. Discrete grid $\implies \Gamma(x)$ and $V(x)$ have finite # of elements
Possible ways to implement VFI

3. Piecewise linear
   - *choices* are no longer constrained to be on grid
   - \( V^i(\cdot) \) is characterized by function values on grid
   - Simply do maximization on grid

4. Regular polynomial
   - *choices* are no longer constrained to be on grid
   - calculate values \( V \) on grid
   - obtain \( V^{i+1} \) by fitting polynomial through calculated point
Convergence

- There are several convergence results for VFI
- Some such results for Euler equation methods
  - but you have to do it right (e.g. use time & not fixed-point iteration)
- But especially for more complex problems, VFI is more likely to converge
Speed; algorithm choice

- VFI: because of the max operator you typically can only iterate
  - slow if discount factor is close to 1

- Euler equation method have more options
  - calculating fixed point directly with equation solver typically faster
Speed; impact choices on V & Euler

VFI tends to be slow in many typical economic applications

- Reason: value function is flat \( \Rightarrow \) hard to find max
  - important to be aware of this
  - Krusell and Smith (1996) show that utility loss of keeping capital stock constant is minor in neoclassical growth model
- But shouldn’t a flat utility function be problematic for Euler eq. methods as well?
Speed; impact choices on V & Euler

Example to show Euler eq. methods less affected by flatness

\[
\max_{x_1, x_2} x_1^{1-\nu} + x_2^{1-\nu}
\]
\[
s.t. \quad x_1 + x_2 \leq 2
\]
\[
\quad x_1, x_2 \geq 0
\]
Consider a *huge* move away from optimum

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$u(1, 1)$</th>
<th>$u(2, 0)$</th>
<th>consumption equivalent loss</th>
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</thead>
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<td>0.01</td>
<td>2</td>
<td>1.9862</td>
<td>0.7%</td>
</tr>
<tr>
<td>0.001</td>
<td>2</td>
<td>1.9986</td>
<td>0.07%</td>
</tr>
</tbody>
</table>
Speed; impact choices on V & Euler

First-order condition:

\[
\left( \frac{x_1}{x_2} \right)^{-\nu} = 1 \text{ or } x_1 = 1^{-1/\nu} \times x_2
\]

Marginal rates of substitution:

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$x_1 = x_2 = 1$</th>
<th>$x_1 = 2, x_2 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1</td>
<td>$\infty$</td>
</tr>
<tr>
<td>0.001</td>
<td>1</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
Dealing with complex problems

- Both VFI and Euler-equation methods can deal with inequality constraints

- Euler equations require first-order conditions to be sufficient
  - this requires concavity (utility function) and convex opportunity set
  - this is not always satisfied
Non-convex problem - example

Environment:

- Two technologies:
  - $y_t = k_t^\alpha$
  - $y_t = Ak_t^\alpha$ with $A > 1$

- Higher-productivity technology can be used after paying a one-time cost $\psi$
Non-convex problem - example

\[ W(k) = \max \left\{ \max_{k+1} k^\alpha - k + 1 + \beta W(k+1), \quad \max_{k+1} k^\alpha - k + 1 - \psi + \beta V(k+1) \right\} \]

\[ V(k) = \max_{k+1} A k^\alpha - k + 1 + \beta V(k+1) \]
RHS Bellman equation for low capital stock (k=0.1)
Ultimate value function

\[ V(K) \]

\[ W(K) \text{ without upgrade} \]

\[ W(K) \text{ with upgrade} \]
References

- Slides on perturbation; available online.
- Slides on projection methods; available online.
  - shows that time-iteration converges even in the presence of inequality constraints