Stability, Multiplicity, and Sunspots (deriving solutions to linearized system & Blanchard-Kahn conditions)

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Content

1. A lot on sunspots
2. A simple way to get policy rules in a linearized framework
    - and an even simpler way based on time iteration (an idea of Pontus Rendahl)

Introduction

- What do we mean with non-unique solutions?
  - multiple solution versus multiple steady states
- What are sunspots?
- Are models with sunspots scientific?

Terminology

- Definitions are very clear
  - (use in practice can be sloppy)

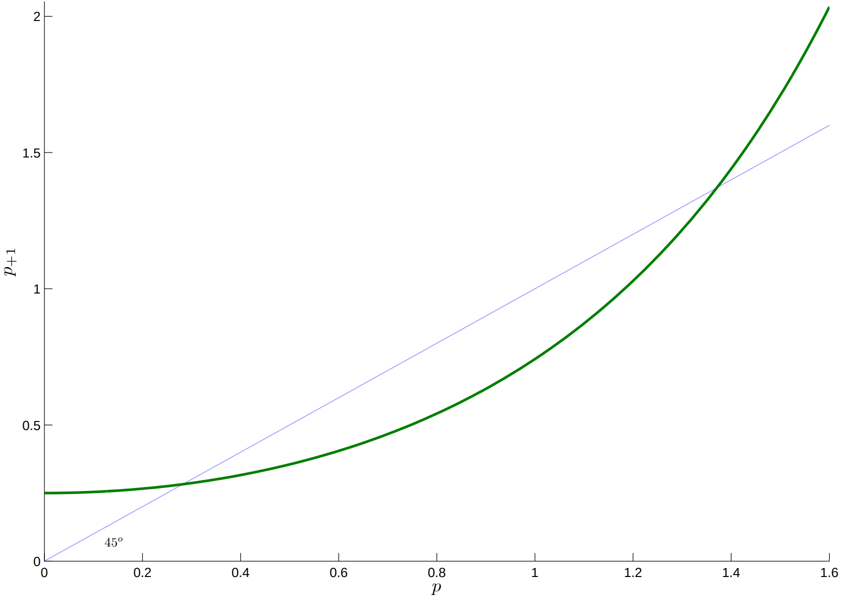
Model:

$$H(p_{+1}, p) = 0$$

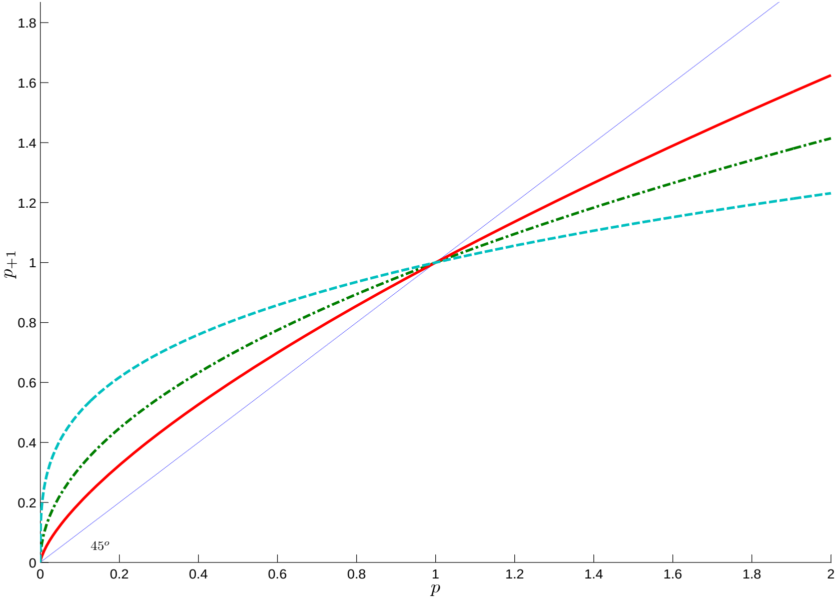
Solution:

$$p_{+1} = f(p)$$

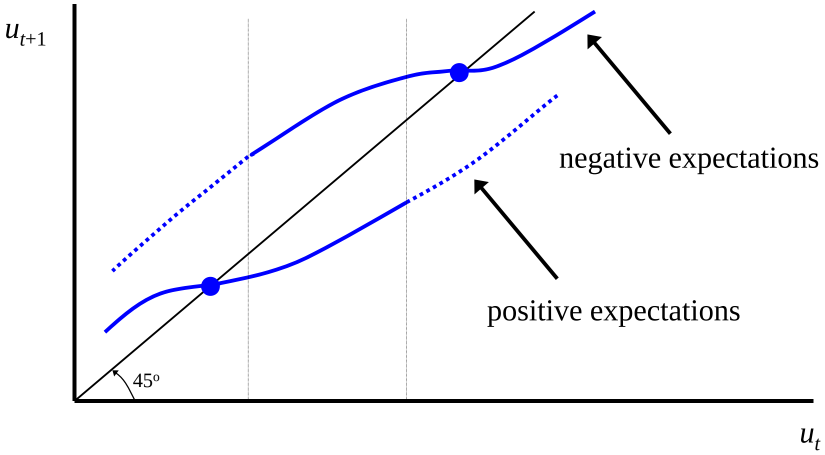
Unique solution & multiple steady states



Multiple solutions & unique (non-zero) steady state



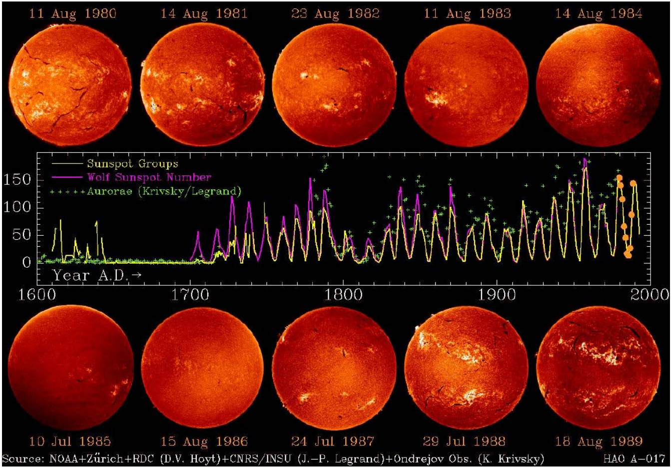
Multiple steady states & sometimes multiple solutions



From Den Haan (2007)

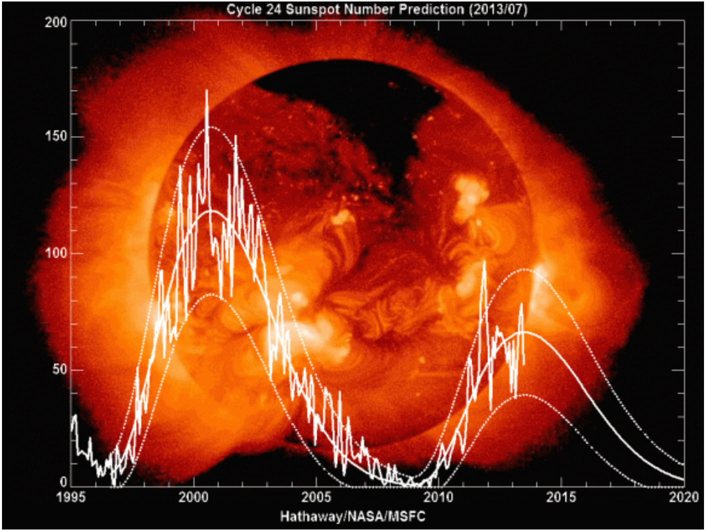
Large sunspots (around 2000 at the peak)

Past Sun Spot Cycles



Sun spots even had a "Great Moderation"

Current cycle (at peak again)



Cute NASA video

- https://www.youtube.com/watch?v=UD5VViT08ME

Sunspots in economics

- **Definition:** a solution is a sunspot solution if it depends on a stochastic variable *from outside system*
- **Model:**

$$
\begin{aligned}
0 &= \mathbb{E}H(p_{t+1}, p_t, d_{t+1}, d_t) \\
d_t &: \text{exogenous random variable}
\end{aligned}
$$

Sunspots in economics (Cass & Shell 1983)

- **Non-sunspot solution:**

$$p_t = f(p_{t-1}, p_{t-2}, \cdots, d_t, d_{t-1}, \cdots)$$

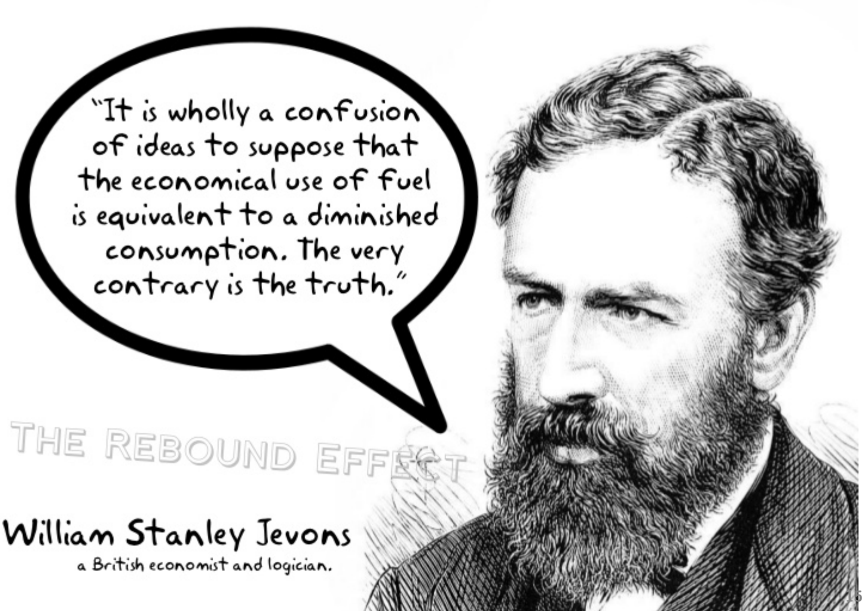
- **Sunspot:**

$$\begin{aligned} p_t &= f(p_{t-1}, p_{t-2}, \cdots, d_t, d_{t-1}, \cdots, s_t) \\ s_t &: \text{random variable with } \mathbb{E}[s_{t+1}] = 0 \end{aligned}$$

Origin of sunspots in economics

- William Stanley Jevons (1835-82) explored empirical relationship between sunspot activity (that is, the real thing!!!) and the price of corn.
- Fortunately, Jevons had some other contributions as well, such as "Jevons Paradox". His work is considered to be the start of mathematical economics.

Jevons Paradox



Sunspots and science

Why are sunspots attractive

- sunspots: s_t matters, just because agents believe this
  - self-fulfilling expectations don't seem that unreasonable
- sunspots provide many sources of shocks
  - number of sizable fundamental shocks small

Sunspots and science

Why are sunspots not so attractive

- Purpose of science is to come up with predictions
  - If there is one sunspot solution, there are zillion others as well
- Support for the conditions that make them happen not overwhelming
  - you need sufficiently large increasing returns to scale or externality

Overview

1. Getting started
    - simple examples
2. General derivation of Blanchard-Kahn solution
    - When unique solution?
    - When multiple solution?
    - When no (stable) solution?
3. When do sunspots occur?
4. Numerical algorithms and sunspots

Getting started

- **Model:** $y_t = \rho y_{t-1}$

Getting started

- $$\textbf{Model:}\quad y_t = \rho y_{t-1}$$

- infinite number of solutions, independent of the value of ρ

Getting started

- **Model:** $$\begin{aligned} y_{t+1} &= \rho y_t \\ y_0 &\text{ is given} \end{aligned}$$

Getting started

- **Model:** $$\begin{array}{l} y_{t+1} = \rho y_t \\ y_0 \text{ is given} \end{array}$$
- unique solution, independent of the value of ρ

Getting started

- Blanchard-Kahn conditions apply to models that add as a requirement that the series do not explode

Model:

$$y_{t+1} = \rho y_t$$

y_t cannot explode

- $\rho > 1$: unique solution, namely $y_t = 0$ for all t
- $\rho < 1$: many solutions
- $\rho = 1$: many solutions
  - be careful with $\rho = 1$, uncertainty matters

State-space representation

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

$$\mathbb{E}\left[\varepsilon_{t+1} | I_t\right] = 0$$

y_t : is an $n \times 1$ vector
$m \leq n$ elements are not determined

some elements of ε_{t+1} are not exogenous shocks but prediction errors

Neoclassical growth model and state space representation

$$\left(\exp(z_t)k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - k_t\right)^{-\gamma} =$$
$$\mathbb{E}\left[\begin{array}{c} \beta\left(\exp(z_{t+1})k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} \\ \times\left(\alpha\exp\left(z_{t+1}\right)k_t^{\alpha-1} + 1 - \delta\right) \end{array}\middle| I_t\right]$$

or equivalently without $\mathbb{E}\left[\cdot\right]$

$$\left(\exp(z_t)k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - k_t\right)^{-\gamma} =$$
$$\beta\left(\exp(z_{t+1})k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma}$$
$$\times\left(\alpha\exp\left(z_{t+1}\right)k_t^{\alpha-1} + 1 - \delta\right)$$
$$+e_{\mathsf{E},t+1}$$

Neoclassical growth model and state space representation

Linearized model:

$$k_{t+1} = a_1 k_t + a_2 k_{t-1} + a_3 z_{t+1} + a_4 z_t + e_{\mathsf{E},t+1}$$
$$z_{t+1} = \rho z_t + e_{z,t+1}$$
$$k_0 \text{ is given}$$

- k_t is end-of-period t capital
  - $\Longrightarrow k_t$ is chosen in t

Neoclassical growth model and state space representation

$$\begin{bmatrix} 1 & 0 & -a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ k_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} -a_1 & -a_2 & -a_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{bmatrix} \begin{bmatrix} k_t \\ k_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathsf{E},t+1} \\ 0 \\ e_{z,t+1} \end{bmatrix}$$

Dynamics of the state-space system

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

$$\begin{aligned} y_{t+1} &= -A^{-1}By_t + A^{-1}\varepsilon_{t+1} \\ &= Dy_t + A^{-1}\varepsilon_{t+1} \end{aligned}$$

Thus

$$y_{t+1} = D^t y_1 + \sum_{l=1}^{t} D^{t-l} A^{-1} \varepsilon_{l+1}$$

Jordan matrix decomposition

$$D = P\Lambda P^{-1}$$

- Λ is a diagonal matrix with the eigen values of D
- without loss of generality assume that $|\lambda_1| \geq |\lambda_2| \geq \cdots |\lambda_n|$

Let

$$P^{-1} = \begin{bmatrix} \tilde{p}_1 \\ \vdots \\ \tilde{p}_n \end{bmatrix}$$

where $\tilde{p}_i$ is a $(1 \times n)$ vector

Dynamics of the state-space system

$$
\begin{aligned}
y_{t+1} &= D^t y_1 + \sum_{l=1}^{t} D^{t-l} A^{-1} \varepsilon_{l+1} \\
&= P \Lambda^t P^{-1} y_1 + \sum_{l=1}^{t} P \Lambda^{t-l} P^{-1} A^{-1} \varepsilon_{l+1}
\end{aligned}
$$

Dynamics of the state-space system

multiplying dynamic state-space system with P^{-1} gives

$$P^{-1}y_{t+1} = \Lambda^t P^{-1} y_1 + \sum_{l=1}^{t} \Lambda^{t-l} P^{-1} A^{-1} \varepsilon_{l+1}$$

or

$$\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t} \lambda_i^{t-l} \tilde{p}_i A^{-1} \varepsilon_{l+1}$$

recall that y_t is $n \times 1$ and $\tilde{p}_i$ is $1 \times n$. Thus, $\tilde{p}_i y_t$ is a scalar

Model

1. $\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t} \lambda_i^{t-l} \tilde{p}_i A^{-1} \varepsilon_{l+1}$
2. $\mathbb{E}\left[\varepsilon_{t+1} | I_t\right] = 0$
3. m elements of y_1 are not determined
4. y_t cannot explode

Reasons for multiplicity

1. There are free elements in y_1
2. The only constraint on $e_{\mathsf{E},t+1}$ is that it is a prediction error.
   - This leaves lots of freedom

Eigen values and multiplicity

- Suppose that $|\lambda_1| > 1$
- To avoid explosive behavior it *must* be the case that

1. $\tilde{p}_1 y_1 = 0$ and
2. $\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

- Simply an additional equation to pin down some of the free elements
- Much better: This is the policy rule in the first period

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

Neoclassical growth model:

- $y_1 = [k_1, k_0, z_1]^T$
- $|\lambda_1| > 1$, $|\lambda_2| < 1$, $\lambda_3 = \rho < 1$
- $\tilde{p}_1 y_1$ pins down k_1 as a function of k_0 and z_1
    - this is the policy function in the first period

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$$

- This pins down $e_{\mathsf{E},t}$ as a function of $\varepsilon_{z,t}$
- That is, the prediction error must be a function of the structural shock, $\varepsilon_{z,t}$, and cannot be a function of other shocks,
    - i.e., there are no sunspots

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$$

Neoclassical growth model:

- $\tilde{p}_1 A^{-1} \varepsilon_t$ says that the prediction error $e_{\mathsf{E},t}$ of period t is a fixed function of the innovation in period t of the exogenous process, $e_{z,t}$

How to think about #1 combined with #2?

$$\tilde{p}_1 y_t = 0 \quad \forall t$$

- Without sunspots
  - i.e. with $\tilde{p}_1 A^{-1} \varepsilon_t = 0 \;\; \forall t$
- k_t is pinned down by k_{t-1} and z_t *in every period*.

Blanchard-Kahn conditions

- Uniqueness: For every free element in y_1, you need one $\lambda_i > 1$
- Multiplicity: Not enough eigenvalues larger than one
- No stable solution: Too many eigenvalues larger than one

How come this is so simple?

- In practice, it is easy to get

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

- How about the next step?

$$y_{t+1} = -A^{-1}By_t + A^{-1}\varepsilon_{t+1}$$

- **Bad news**: A is often not invertible
- **Good news**: Same set of results can be derived
  - Schur decomposition (See Klein 2000 and Soderlind 1999)

How to check in Dynare

Use the following command after the model & initial conditions part

```
check;
```

Example - x predetermined - 1st order

$$
\begin{aligned}
x_{t-1} &= \phi x_t + \mathbb{E}_t \left[z_{t+1} \right] \\
z_t &= 0.9 z_{t-1} + \varepsilon_t
\end{aligned}
$$

- $|\phi| > 1$: Unique stable fixed point
- $|\phi| < 1$: No stable solutions; too many eigenvalues > 1

Example - x predetermined - 1st order

Corresponding state space system:

$$\begin{bmatrix} \phi & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_{t+1} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1/\phi & 0 \\ 0 & 0.9 \end{bmatrix}$$

- No sunspots, since $\mathbb{E}_t \left[z_{t+1} \right]$ is the only expectation.
- No multiplicity because of free initial conditions either one starting value for x given.
- So we just need stability $\implies |\phi| > 1$

Example - x predetermined - 2nd order

$$\begin{aligned} \phi_2 x_{t-1} &= \mathbb{E}_t\left[\phi_1 x_t + x_{t+1} + z_{t+1}\right] \\ z_t &= 0.9 z_{t-1} + \varepsilon_t \end{aligned}$$

- $\phi_1 = -2.25$, $\phi_2 = -0.5$: Unique stable fixed point
  $(1 + \phi_1 L - \phi_2 L^2) x_t = (1 - 2L)(1 - \frac{1}{4}L) x_t$
- $\phi_1 = -3.5$, $\phi_2 = -3$: No stable solution; too many eigenvalues > 1
  $(1 + \phi_1 L - \phi_2 L^2) x_t = (1 - 2L)(1 - 1.5L) x_t$
- $\phi_1 = -1$, $\phi_2 = -0.25$: Multiple stable solutions; too few eigenvalues > 1
  $(1 + \phi_1 L - \phi_2 L^2) x_t = (1 - 0.5L)(1 - 0.5L) x_t$

Example - x predetermined - 2nd order

Corresponding state space system:

$$\begin{bmatrix} 1 & \phi_1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ x_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & -\phi_2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -0.9 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathbb{E},t+} \\ 0 \\ \varepsilon_t \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

- The Λ matrix is for the first numerical example ($\phi_1 = -2.25$, $\phi_2 = -0.5$)

Example - x not predetermined - 1st order

$$
\begin{aligned}
x_t &= \mathbb{E}_t \left[\phi x_{t+1} + z_{t+1}\right] \\
z_t &= 0.9 z_{t-1} + \varepsilon_t
\end{aligned}
$$

- $|\phi| < 1$: Unique stable fixed point
- $|\phi| > 1$: Multiple stable solutions; too few eigenvalues > 1

Example - x not predetermined - 1st order

Corresponding state space system:

$$\begin{bmatrix} \phi & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathbb{E},t+1} \\ \varepsilon_{t+1} \end{bmatrix}$$
$$\Lambda = \begin{bmatrix} \phi & 0 \\ 0 & 0.9 \end{bmatrix}$$

- No sunspots, since $\mathbb{E}_t\left[z_{t+1}\right]$ is the only expectation.
- No multiplicity because of free initial conditions either one starting value for x given.
- So we just need stability $\Longrightarrow |\phi| > 1$

Solutions to linear systems

1. The analysis outlined above
(requires A to be invertible)
2. Generalized version of analysis above
(see Klein 2000)

3. Apply time iteration to linearized system
(I learned this from Pontus Rendahl)

Solutions to linear systems

Model:

$$\Gamma_2 k_{t+1} + \Gamma_1 k_t + \Gamma_0 k_{t-1} = 0$$

or

$$\begin{bmatrix} \Gamma_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ k_t \end{bmatrix} + \begin{bmatrix} \Gamma_1 & \Gamma_0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} k_t \\ k_{t-1} \end{bmatrix} = 0$$

Standard approaches

1. The method outlined above
$\Longrightarrow$ a unique solution of the form

$$k_t = ak_{t-1}$$

if BK conditions are satisfied

2. Impose that the solution is of the form

$$k_t = ak_{t-1}$$

and solve for a from

$$\Gamma_2 a^2 k_{t-1} + \Gamma_1 a k_{t-1} + \Gamma_0 k_{t-1} = 0 \quad \forall k_{t-1}$$

Time iteration

- Impose that the solution is of the form

$$k_t = ak_{t-1}$$

- Use time iteration scheme, starting with $a_{[1]}$
- Recall that time iteration means using the guess for *tomorrow*s behavior and then solve for *today*s behavior

(This simple procedure was pointed out to me by Pontus Rendahl)

Time iteration

- Follow the following iteration scheme, starting with $a_{[1]}$
  - Use $a_{[i]}$ to describe next period's behavior. That is,

$$\Gamma_2 a_{[i]} k_t + \Gamma_1 k_t + \Gamma_0 k_{t-1} = 0$$

(note the difference with last approach on previous slide)

  - Obtain $a_{[i+1]}$ from

$$(\Gamma_2 a_{[i]} + \Gamma_1) k_t + \Gamma_0 k_{t-1} = 0$$

$$k_t = -\left(\Gamma_2 a_{[i]} + \Gamma_1\right)^{-1} \Gamma_0 k_{t-1}$$

$$a_{[i+1]} = -\left(\Gamma_2 a_{[i]} + \Gamma_1\right)^{-1} \Gamma_0$$

Advantages of time iteration

- It is simple even if the "A matrix" is not invertible.
  (the inversion required by time iteration seems less problematic in practice)
- Since time iteration is linked to value function iteration, it has nice convergence properties

Example

$$k_{t+1} - 2k_t + 0.75k_{t-1} = 0$$

- The two solutions are

$$k_t = 0.5k_{t-1} \ \& \ k_t = 1.5k_{t-1}$$

- Time iteration on $k_t = a_{[i]}k_{t-1}$ converges to stable solution for all initial values of $a_{[i]}$ except 1.5.

References and Acknowledgements

- Larry Christiano taught me (a long time ago) this simple way of deriving the BK conditions and I think that I did not even change the notation.

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  - simple model in which the size of the shocks has long-term consequences
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- Klein, Paul, 2000, Using the Generalized Schur form to Solve a Multivariate Linear Rational Expectations Model, Journal of Economic Dynamics and Control, 1405-1423.
  - in case you want to do the analysis without the simplifying assumption that A is invertible
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  - also doesn't assume that A is invertible; possibly a more accessible paper